

LIFTABLE DERIVATIONS FOR GENERICALLY SEPARABLY ALGEBRAIC MORPHISMS OF SCHEMES

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ABSTRACT. We consider dominant, generically algebraic (e.g. generically finite), and tamely ramified (if the characteristic is positive) morphisms $\pi : X/S \rightarrow Y/S$ of S -schemes, where Y, S are Noetherian and integral and X is a Krull scheme (e.g. normal Noetherian), and study the sheaf of tangent vector fields on Y that lift to tangent vector fields on X . We give an easily computable description of these vector fields using valuations along the critical locus. We apply this to answer the question when the liftable derivations can be defined by a tangency condition along the discriminant. In particular, if π is a blow-up of a coherent ideal I we show that tangent vector fields that preserve the Ratliff-Rush ideal (equals $[I^{n+1} : I^n]$ for high n) associated to I are liftable, and that all liftable tangent vector fields preserve the integral closure of I . We also generalise to positive characteristic Seidenberg's theorem that all tangent vector fields can be lifted to the normalisation, assuming tame ramification.

INTRODUCTION

Let $\pi : X/S \rightarrow Y/S$ be a dominant morphism of S -schemes, where Y and S are Noetherian and X is a Krull scheme (e.g. Noetherian and normal), so in particular X and Y are integral. Consider the diagram

$$\begin{array}{ccccccc} T_{X/S} & \xrightarrow{d\pi} & T_{X/S \rightarrow Y/S} & \longrightarrow & \mathcal{C}_{X/Y} & \longrightarrow & 0 \\ & & \uparrow \psi & & & & \\ & & \pi^{-1}(T_{X/S}) & & & & \end{array}$$

where $T_{X/S} = \text{Hom}_{\mathcal{O}_X}(\Omega_{X/S}, \mathcal{O}_X)$ is the sheaf of S -relative tangent vector fields, $d\pi$ is the tangent morphism and ψ is the canonical morphism to the sheaf $T_{X/S \rightarrow Y/S}$ of derivations from $\pi^{-1}(\mathcal{O}_Y)$ to \mathcal{O}_X ; if Y/S is locally of finite type and either π is flat or Y/S is smooth, then $T_{X/S \rightarrow Y/S} = \pi^*(T_{Y/S})$. The critical locus C_π is the support of the critical module $\mathcal{C}_{X/Y}$, and the discriminant set D_π is the closure of $\pi(C_\pi)$. Say that a section ∂ of $T_{Y/S}$ is liftable if there exists a section $\bar{\partial}$ of $T_{X/S}$ such that $d\pi(\bar{\partial}) = \psi(\partial)$ (suppressing domains of definition of ∂ and $\bar{\partial}$). We denote the sheaf of liftable tangent vector fields by $T_{Y/S}^\pi$.

Let $T_{Y/S}(\log^\pi I_{D_\pi}) = T_{Y/S}(\log^\pi D_\pi)$ be the sheaf of π -logarithmic vector fields. This sheaf is easily computed using the discrete valuations ν_1, \dots, ν_r of the function field of X defined by order of vanishing along the components of the critical divisor locus $C_\pi = C_1 + C_2 + \dots + C_r$. A section ∂ in $T_{Y/S}$ is π -logarithmic, or logarithmic along the morphism π , if it does not lower the value for the discrete valuations at the generic points of C_π of local generators of the ideal I_{D_π} , at points below the generic points, i.e. $\nu_i(\partial(f_j)) \geq \nu_i(f_j)$ when $f_j \in I_i$, where I_i is any defining ideal of the discriminant D_π at the generic point of $\pi(C_i)$. The sheaf of *logarithmic*

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vector fields along a subscheme defined by an ideal I , denoted $T_{Y/S}(I)$, is the sheaf of tangent vector fields ∂ that preserve the ideal I , $\partial(I) \subset I$. We describe the liftable tangent vector fields when π is generically algebraic (the induced extension of function fields is algebraic), and tamely ramified (Def. 1.5.5).

Theorem 2.2.1. *Let $\pi : X/S \rightarrow Y/S$ be a morphism of schemes over a scheme S . Assume that Y and S are Noetherian and X is a Krull scheme. Assume that π is dominant, generically algebraic and tamely ramified. If Y is of unequal characteristic, assume that Y is absolutely tamely ramified. Let D_π be the discriminant set of π . Then:*

(L_1) *Let J_{D_π} be any defining ideal of D_π . Then*

$$T_{Y/S}^\pi = T_{Y/S}(\log^\pi J_{D_\pi}).$$

(L_2) *If the residue field extension $k_c/k_{\pi(c)}$ is algebraic for each point $c \in C_\pi$ of height 1, then*

$$T_{Y/S}^\pi = T_{Y/S}(I_{D_\pi}),$$

where I_{D_π} is the (reduced) ideal of the closure of the discriminant set D_π .

(L_3) *Let I_{C_π} be the ideal of the critical locus of π . Then $\pi^{-1}(T_{Y/S}^\pi) \subset T_{X/S}(I_{C_\pi})$, i.e. the liftable vector fields are tangent to the critical locus.*

We give concrete examples at the end of the paper.

Here (L_1) is the main new content of Theorem 2.2.1 while (L_2) is a generalisation of earlier work, but then usually formulated for affine schemes. The first result of this type is due to Zariski who proved the following [36, § 5, Th. 2]. Let R be an integrally closed local Nagata ring, with quotient field $K(R)$. Let R' be the integral closure in a finite separable extension L of $K(R)$ (thus R' is finite over R). Let $D \subset \text{Spec } R$ be the set of ramified primes of height 1 in R , and assume that each prime in D is tamely ramified. Let ∂ be a derivation of R such that for each point $x \in D$ of height 1 there exists an element $r \in \mathfrak{m}_x \setminus \mathfrak{m}_x^2$ such that $\partial(r) = 0$. It then follows that $\partial(R') \subset R'$. According to Theorem 2.2.1, with $X = \text{Spec } R'$ and $Y = \text{Spec } R$, knowing the existence of the element r is not necessary, as is the assumption that R be a Nagata ring; the sufficient and necessary condition is (L_2). Scheja and Storch [29] proved (L_2) when R contains the rational numbers, assuming the R -module $R\partial(R)$ be of finite type. We see that this assumption is not needed and we also get the assertion in positive characteristic when R/A is tamely ramified.

A dominant morphism π is *weakly submersive* if $T_{Y/S}^\pi = T_{Y/S}$, i.e. all tangent vector fields lift to vector fields on X . If Y/S is non-smooth and π is non-flat this is *not* the same as surjective tangent morphism $d\pi$, in which case we say π is submersive. Seidenberg [30] proved that the normalisation morphism of a Noetherian integral scheme X/S is weakly submersive when S is defined over the rational numbers. We generalise this to positive characteristic (Th. 2.3.1), requiring that the normalisation morphism be tamely ramified. The proof is by reducing to the case when the normalisation is a discrete valuation ring, which is quite different from Seidenberg's proof (see Remark 2.3.4). In a forthcoming paper we will prove that the constructive resolutions of singularities of a variety X/k ($\text{Char } k = 0$) presented in [4, 8, 35] are weakly submersive. This also gives the result that the multiplier ideal $J(\alpha)$ (discussed in [20]) of a coherent ideal I is preserved by derivations that preserve I , i.e. $T_{X/k}(I) \subset T_{X/k}(J(\alpha))$.

Theorem 2.2.1 describes the liftable derivations at generic points of the discriminant D_π in terms of a set of discrete valuations associated to the critical locus. A

natural question is whether the liftable tangent fields coincide with the vector fields that are tangent to some subscheme whose underlying space is D_π . More precisely, does there exist a coherent defining ideal I of D_π satisfying

$$(*) \quad T_{Y/S}^\pi = T_{Y/S}(I)?$$

The best one can hope for, second to weakly submersive, is that all vector fields that are tangent to the discriminant can be lifted, i.e. the above equality holds when I is the (radical) ideal of D_π . We call such morphisms *differentially ramified*, and know already from Theorem 2.2.1 that residually algebraic tamely ramified morphisms are differentially ramified. A stronger notion is that π be *uniformly ramified*, meaning that at each generic point ξ of the critical locus C_π , the stalk $I_{D_\pi, \pi(\xi)}$ has a basis with constant value in the valuation ring $\mathcal{O}_{X, \xi}$. We prove in Theorem 3.1.3 that uniformly ramified morphisms are differentially ramified, and if for each generic point $\xi \in C_\pi$ there exists a basis $\{x_1, \dots, x_r\}$ of $I_{D, \pi(\xi)}$ and derivations $\{\partial_1, \dots, \partial_r\} \subset T_{Y/S, \pi(\xi)}$ such that the matrix $(\partial_j(x_i))$ is invertible, then the converse also holds.

It is particularly interesting to characterise the liftable tangent fields for birational morphisms, forming an important class of generically separably algebraic morphisms. In the light of (L_2) and (L_3) in Theorem 2.2.1 this also characterises the liftable vector fields for “alterations”, i.e. compositions of finite and birational morphisms. We have already discussed certain birational morphisms, but now consider any (projective) birational morphism of integral Noetherian schemes, which we know always is the blow-up of some fractional ideal. Hence let $\pi : Bl_I(X) \rightarrow X$ be the blow-up of a given fractional ideal I on X . Here $Bl_I(X)$ need not be Krull (normal) so Theorem 2.2.1 is not directly applicable, but we can apply it to attain a lower and upper inclusion for $T_{X/S}^\pi$ by sheaves of vector fields that are tangent to certain subschemes with the same underlying space as the discriminant. One first easily gets that the sub-Lie algebroid of vector fields that preserve I are liftable, $T_{X/S}(I) \subset T_{X/S}^\pi$, but in general this inclusion is strict. There is also a latitude in the choice of I , for different ideals may give the same blow-up, and we ask if there is a choice \tilde{I} such that $Bl_I(X) \cong Bl_{\tilde{I}}(X)$ (isomorphism over X) and $T_{X/S}^\pi = T_{X/S}(\tilde{I})$? Let $\hat{I} = \cup_{n \geq 1} [I^{n+1} : I^n]$ be the Ratliff-Rush ideal associated to I [27] and \bar{I} its integral closure; then $I \subset \hat{I} \subset \bar{I}$. In Theorem 3.2.2 we prove

$$T_{X/S}(\hat{I}) \subset T_{X/S}^\pi \subset T_{X/S}(\bar{I}),$$

where the right-hand inclusion holds under the additional assumption that the normalisation morphism of $Bl_I(X)$ be tamely ramified (make this assumption in this paragraph). We therefore succeed in a positive answer to our question $(*)$ above when the Ratliff-Rush ideal \hat{I} associated to I is integrally closed, getting $T_{X/S}^\pi = T_{X/S}(\hat{I})$, since $Bl_{\hat{I}}(X)$ is isomorphic to $Bl_I(X)$. An immediate consequence is that blow-ups of radical ideals are differentially ramified. Combining with Theorem 3.1.3 one also gets this: Let $\pi : \tilde{X}/S \rightarrow X/S$ be a tamely ramified blow-up of a reduced subscheme V/S where \tilde{X} is assumed to be Krull and X/S is a Noetherian integral scheme which is smooth at the generic points of V , then π is uniformly ramified (Cor. 3.2.3).

One should note that several results in this paper have straightforward holomorphic counterparts. For instance, for a finite holomorphic map $\pi : \mathbf{C}^m \rightarrow \mathbf{C}^m$ we have $T_{\mathbf{C}^m}^\pi = T_{\mathbf{C}^m}(I_{D_\pi})$, which was proven in a different way in [2].

We also mention how liftable tangent vector fields can be used. One may interpret $T_{Y/S}^\pi$ as the sheaf of infinitesimal symmetries of π , so that the fibres of liftable tangent vector fields correspond to directions in the base where the fibres of π do not deform. Thus a good understanding of $T_{Y/S}^\pi$ is useful for the

study of deformations in a (flat) family of schemes. We tie this up with Zariski's notion of analytic equisingularity stratification of a hypersurface X in \mathbf{C}^n . The dominant stratum is where X is smooth, so that smooth points do not belong to the critical locus of the restriction $\pi = p|_X : X \rightarrow \mathbf{C}^{n-1}$ for *generic* projections $p : \mathbf{C}^n \rightarrow \mathbf{C}^{n-1}$. Hence all tangent vector fields near $\pi(x)$ lift to vector fields near a smooth point x ; therefore, by Zariski's lemma [23, Corollary to Th. 30.1], all smooth points on X can be regarded equisingular, since they have isomorphic analytic neighbourhoods. Of course, this must be so since π is étale at x , so \mathcal{O}_x is analytically isomorphic to the ring of formal power series (or the analytic localisation of \mathcal{O}_x is isomorphic to the ring of convergent power series in $n - 1$ variables). The next stratum consists of points x where the discriminant of π_x is non-empty but smooth for generic p , so that, at $\pi(x) \in D_\pi$, the stalk $T_{D_\pi, \pi(x)}$ of tangent vector fields on D_π near $\pi(x)$ has a basis of non-vanishing tangent vector fields, which extend to non-vanishing vector fields near $\pi(x)$ in \mathbf{C}^{n-1} tangent to D_π , hence they are liftable. Again by Zariski's lemma, points with smooth discriminants (and generic p) can be regarded equisingular, since they have isomorphic analytic neighbourhoods. If the generic discriminants D_π are not smooth, by induction in the dimension n the equisingularity stratification of D_π is defined, which can be pulled back to get a stratification of X . For a more complete discussion of Zariski's equisingularity stratification see [21, 34]. Another situation where a good description of $T_{Y/S}^\pi$ is useful, is for describing direct images of D-modules (see [5]) with respect to a morphism π of complex algebraic manifolds, say the zeroth direct image of the structure sheaf $R^0\pi_+(\mathcal{O}_X) = \pi_*(\omega_{X/Y} \otimes_{\mathcal{D}_X} \pi^*(\mathcal{D}_Y))$ (in the complement of D_π it is a Gauss-Manin connection), where \mathcal{D}_X and \mathcal{D}_Y are the rings of differential operators on X and Y , and $\omega_{X/Y}$ is the relative canonical bundle. Then we have a surjective homomorphism of \mathcal{D}_Y -modules

$$\frac{\mathcal{D}_Y}{\mathcal{D}_Y T_Y^\pi} \rightarrow R^0\pi_+(\mathcal{O}_X) \rightarrow 0,$$

which in particular gives a bound on the singular support of $R^0\pi_+(\mathcal{O}_X)$.

Standard notions. Let A and R be local rings (A, \mathfrak{m}_A, k_A) and (R, \mathfrak{m}_R, k_R) , where k_A, k_R are the residue fields, and $\pi : A \rightarrow R$ a homomorphism of rings. Then π is local if $\pi(\mathfrak{m}_A) \subset \mathfrak{m}_R$, R dominates A if $\pi^{-1}(\mathfrak{m}_R) = \mathfrak{m}_A$, and π is (separably) algebraic if the extension of fraction fields $K(R)/K(A)$ is (separably) algebraic, residually algebraic/finite/separable if k_R/k_A is algebraic/finite/separable; in particular, R/A is residually finite if it is quasi-finite, while the converse holds if $R\mathfrak{m}_A$ is an ideal of definition of R . We refer to [13, 23] for the basic results about formally smooth/unramified morphisms; the discrete topology is intended if no other topology is mentioned. A morphism of schemes $\pi : X \rightarrow Y$ is dominant if the morphism $\pi^{-1}(\mathcal{O}_Y) \rightarrow \mathcal{O}_X$ is injective; π is generically (separably) algebraic if π is dominant, X and Y are reduced, and for points $x \in X$ that map to a generic point $\xi \in Y$ the corresponding residue field extension k_x/k_ξ is (separably) algebraic, and in particular formally étale. As common practice, by \mathcal{O}_X -module we mean a sheaf of modules, and by writing $m \in M$ for a sheaf M we mean that m is a section of M over a suitably defined open set. By a *Lie algebroid* on X/S is intended an \mathcal{O}_X -module $\mathfrak{g}_{X/S}$ which moreover is a Lie algebra, provided with a homomorphism (as Lie algebras and \mathcal{O}_X -modules) to the tangent sheaf $\alpha : \mathfrak{g}_{X/S} \rightarrow T_{X/S}$, with the obvious compatibility relations $[\delta, f\eta] = \alpha(\delta)(f)\eta + f[\delta, \eta]$, $\delta, \eta \in \mathfrak{g}_{X/S}$, $f \in \mathcal{O}_X$ (see e.g. [19]). We shall only have occasions to study Lie algebroids where α is injective. For example, the sub-sheaf of liftable derivations $T_{Y/S}^\pi$ is a sub-Lie algebroid of $T_{Y/S}$; other sub-Lie algebroids arise from (fractional) ideals I of \mathcal{O}_Y , as the subsheaf $T_{Y/S}(I)$ of $T_{Y/S}$ of derivations ∂ that preserve I , $\partial(I) \subset I$.

1. TANGENT MORPHISMS, LOGARITHMIC DERIVATIONS AND RAMIFICATION

The purpose of the material in Sections 1.4 and 1.5, which is not the minimal necessary to prove our main results, is to provide a useful collection of basic algebraicity/finiteness facts and some ramification theory, with extensions of well-known results, and relate these classical notions to our lifting problem. For instance, we recall concisely the relation between the property that a morphism $A \rightarrow R$ be finite and quasi-finite, respectively, (Prop. 1.4.1) (a formulation of Zariski's main theorem), and the relation between the property that $K(R)/K(A)$ and k_R/k_A be algebraic, respectively, (Prop. 1.4.3). We say a homomorphism of local Noetherian rings $A \rightarrow R$ is ramified if the cotangent mapping $k_R \otimes \mathfrak{m}_A/\mathfrak{m}_A^2 \rightarrow \mathfrak{m}_R/\mathfrak{m}_R^2$ is not injective. This notion of ramification is adequate for the study of derivations, and in the important case when R is a discrete valuation ring this is close to, but stronger than, formally unramified; we describe what ramification means for the liftable derivations (Prop. 1.5.1). Formally unramified/étale morphisms are described in some detail, complementing the literature: Proposition 1.5.2 is a characterisation of separably algebraic field extensions in terms of vanishing differentials, and Theorem 1.5.4 characterises formally unramified morphisms of Noetherian rings R/A as morphisms such that k_R/k_A is separably algebraic and $R\mathfrak{m}_A = \mathfrak{m}_R$, removing a finiteness assumption in Auslander and Buchsbaum's proof of the assertion [3]. Lemma 1.5.8 states that separably algebraic field extensions k'/k extend to \mathfrak{m} -étale extensions of complete p -rings (although not étale in the discrete topology); this is needed in the proof of Theorem 2.2.1 to handle unequal characteristics.

1.1. The tangent morphism. A morphism of S -schemes $\pi : X/S \rightarrow Y/S$ has its canonical exact sequence of differentials

$$(1.1) \quad \pi^*(\Omega_{Y/S}) \rightarrow \Omega_{X/S} \rightarrow \Omega_{X/Y} \rightarrow 0$$

which induces a homomorphism of \mathcal{O}_Y -modules, the *tangent morphism* of π ,

$$d\pi : T_{X/S} = \text{Hom}_{\mathcal{O}_X}(\Omega_{X/S}, \mathcal{O}_X) \rightarrow T_{X/S \rightarrow Y/S},$$

where the \mathcal{O}_X -module of 'derivations from \mathcal{O}_Y to \mathcal{O}_X ' is

$$T_{X/S \rightarrow Y/S} = \text{Hom}_{\mathcal{O}_X}(\pi^*(\Omega_{Y/S}), \mathcal{O}_X) = \text{Hom}_{\pi^{-1}(\mathcal{O}_Y)}(\pi^{-1}(\Omega_{Y/S}), \mathcal{O}_X).$$

There are canonical homomorphisms:

$$(1.2) \quad \psi_0 : \pi^{-1}(T_{Y/S}) \rightarrow T_{X/S \rightarrow Y/S},$$

$$(1.3) \quad \psi : \pi^*(T_{Y/S}) \rightarrow T_{X/S \rightarrow Y/S}.$$

If π is dominant, then ψ_0 is injective, but ψ need be neither surjective nor injective when Y/S is non-smooth.

Example 1.1.1. (i) Let $X = \text{Spec } k[t]$, $Y = \text{Spec } k[t^2, t^3]$, and $S = \text{Spec } k$. Then $T_{X/k} = \mathcal{O}_X \partial_t$ and $T_{Y/k} = \mathcal{O}_Y t \nabla + \mathcal{O}_Y \nabla$ where $\nabla = t \partial_t$. Let π be the morphism of schemes induced by the inclusion of rings. Then $\partial_t \in T_{X/S}$, and $t \otimes \nabla - 1 \otimes t \nabla \in \pi^*(T_Y)$ is non-zero, while $\psi(t \otimes \nabla - 1 \otimes t \nabla) = 0$. Also $d\pi(\partial_t)$ is the non-zero section of $T_{X/S \rightarrow Y/S}$ that is induced by the derivation $\partial_t : k[t^2, t^3] \rightarrow k[t]$; hence $d\pi(\partial_t) \notin \text{Im}(\psi) = \mathcal{O}_Y \nabla$. So ψ is neither injective nor surjective. Here $d\pi$ is injective.

(ii) (Immersion of singular locus) Assume that X/k is a variety of characteristic 0, with singular locus Z ; let I_Z be the ideal of Z . Let $\pi : Z \rightarrow X$ of the inclusion of the locus of non-smooth points and $T_{X/k}(\mathcal{O}_X, I_Z)$ the subsheaf of $T_{X/k}$ consisting of derivations that send \mathcal{O}_X to I_Z . We have an exact sequence

$$0 \rightarrow \pi^*(T_{X/k}(\mathcal{O}_X, I_Z)) \rightarrow \pi^*(T_{X/k}) \xrightarrow{\psi} T_{Z/k \rightarrow X/k} \rightarrow \mathcal{F}_{Z/X} \rightarrow 0$$

where the cokernel $\mathcal{F}_{Z/X}$ of ψ is a subsheaf of $\text{Ext}_{\mathcal{O}_X}^1(\Omega_{X/k}, I_Z)$. The support of $\mathcal{F}_{Z/X}$ consists of points where X is not equisingular, in the strong sense that there exist tangent vector fields on Z that cannot be extended to tangent vector fields on X . Since $T_{X/k} = T_{X/k}(I_Z)$ (see e.g. [19]) there exists a natural homomorphism $\text{can} : \pi^*(T_{X/k}) \rightarrow T_{Z/k}$, and a commutative diagram:

$$\begin{array}{ccc} \pi^*(T_{X/k}) & \xrightarrow{\psi} & T_{Z/k \rightarrow X/k} \\ & \searrow \text{can} & \uparrow d\pi \\ & & T_{Z/k} \end{array}$$

This gives a surjective homomorphism $\mathcal{F}_{Z/X} \rightarrow \mathcal{C}_{Z/X}$. Therefore $d\pi$ is surjective if ψ is surjective.

Proposition 1.1.2. *Assume that Y/S is locally of finite type. The homomorphism ψ is an isomorphism in the following cases:*

- (1) Y/S is smooth.
- (2) π is flat.

Proof. The map ψ is an isomorphism if it induces an isomorphism of stalks, and since $\Omega_{Y/S}$ is coherent and $\Omega_{Y/S,y} \cong \Omega_{\mathcal{O}_{Y,y}/\mathcal{O}_{S,s}}$ (if $U \subset Y$ is affine and $y \in U$, then the canonical mapping $\mathcal{O}_Y(U) \rightarrow \mathcal{O}_{Y,y}$ is étale; then apply the first fundamental exact sequence for differentials)

$$\text{Hom}_{\mathcal{O}_Y}(\Omega_{Y/S}, \mathcal{O}_Y)_y = \text{Hom}_{\mathcal{O}_{Y,y}}(\Omega_{\mathcal{O}_{Y,y}/\mathcal{O}_{S,s}}, \mathcal{O}_{Y,y}),$$

where s is the image of y in S , the condition is that the canonical morphism

$$\psi_x : \mathcal{O}_x \otimes_{\mathcal{O}_y} \text{Hom}_{\mathcal{O}_y}(\Omega_{\mathcal{O}_y/\mathcal{O}_s}, \mathcal{O}_y) \rightarrow \text{Hom}_{\mathcal{O}_x}(\mathcal{O}_x \otimes_{\mathcal{O}_y} \Omega_{\mathcal{O}_y/\mathcal{O}_s}, \mathcal{O}_x)$$

be an isomorphism, where $\mathcal{O}_y \rightarrow \mathcal{O}_x$ is the homomorphism of local rings defined by π and points $x \in X$, $y = \pi(x) \in Y$. By (1) $\Omega_{\mathcal{O}_y/\mathcal{O}_s}$ is free of finite rank over \mathcal{O}_y [14, Prop. 17.2.3], implying the assertion. Since $\Omega_{\mathcal{O}_y/\mathcal{O}_s}$ is of finite presentation, (2) implies the assertion by [23, Th. 7.11]. \square

When ψ is an isomorphism one gets the “ordinary” tangent homomorphism

$$\psi^{-1} \circ d\pi : T_{X/S} \rightarrow \pi^*(T_{Y/S}).$$

If ψ is not an isomorphism one can take the fibre product to get a restricted tangent sheaf

$$\begin{aligned} T_{X/S}^r &= \pi^*(T_{Y/S}) \times_{T_{X/S \rightarrow Y/S}} T_{X/S} \\ &= \{(\delta, \partial) \in \pi^*(T_{X/S}) \times_{T_{X/S \rightarrow Y/S}} T_{X/S} : \psi(\delta) = d\pi(\partial)\}, \end{aligned}$$

so the projection on the first factor $T_{X/S}^r \rightarrow \pi^*(T_{Y/S})$ can also play the role of ‘tangent morphism’. The sheaf $T_{X/S}^r$ is an \mathcal{O}_Y -Lie algebroid (namely the pull-back of the Lie algebroid $T_{Y/S}$) containing the Lie sub-algebroid $T_{X/Y} = \text{Ker}(d\pi)$ of relative tangent vector fields and also the \mathcal{O}_X -Lie algebra $\mathfrak{b}_X = \text{Ker}(\psi)$. One gets

a commutative diagram of \mathcal{O}_X -modules:

$$(1.4) \quad \begin{array}{ccccccc} & & 0 & & 0 & & 0 \\ & & \uparrow & & \uparrow & & \uparrow \\ 0 & \longrightarrow & N_X^4 & \longrightarrow & \mathcal{C}_{X/Y} & \longrightarrow & N_X^3 \longrightarrow 0 \\ & & \uparrow & & \uparrow & & \uparrow \\ 0 & \longrightarrow & \mathfrak{b}_X & \longrightarrow & \pi^*(T_{Y/S}) & \xrightarrow{\psi} & T_{X/S \rightarrow Y/S} \longrightarrow 0 \\ & & \parallel & & \uparrow & & \uparrow \\ 0 & \longrightarrow & \mathfrak{b}_X & \longrightarrow & T_{X/S}^r & \longrightarrow & T_{X/S} \longrightarrow 0 \\ & & & & \uparrow & & \uparrow \\ & & & & T_{X/Y} & \xlongequal{\quad} & T_{X/Y} \\ & & & & \uparrow & & \uparrow \\ & & & & 0 & & 0 \end{array}$$

1.2. Submersions and weak submersions. Consider the vertical sequence above, containing $d\pi$. The stalk at $x \in X$ of the tangent morphism

$$d\pi_x : T_{X/S,x} \rightarrow T_{X/S \rightarrow Y/S,x}$$

induces a map of fibres $d\pi_x : k_x \otimes_{\mathcal{O}_x} T_{X/S,x} \rightarrow k_x \otimes_{\mathcal{O}_x} T_{X/S \rightarrow Y/S,x}$; if $T_{X/S \rightarrow Y/S} \cong \pi^*(T_{X/S})$ (Prop. 1.1.2) one gets the “ordinary” map of tangent spaces $k_x \otimes_{\mathcal{O}_x} T_{X/S,x} \rightarrow k_x \otimes_{\mathcal{O}_y} T_{Y/S,y}$. Say that π is *submersive* at x if $d\pi_x$ is surjective, i.e. $\mathcal{C}_{X/Y,x} = 0$. If $\mathcal{C}_{X/Y,x}$ is of finite type, by Nakayama’s lemma $d\pi_x$ is surjective if the map $d\pi_x$ is surjective. The relation between the critical locus $C_\pi = \text{supp } \mathcal{C}_{X/Y}$ and the ramification locus of a morphism is studied in [18].

There are canonical homomorphisms of \mathcal{O}_Y -modules $\pi_*(d\pi) : \pi_*(T_{X/S}) \rightarrow \pi_*(T_{X/S \rightarrow Y/S})$ and $\pi_*(\psi_0) : T_{Y/S} \rightarrow \pi_*(T_{X/S \rightarrow Y/S})$ so one can form the fibre product

$$\pi_*(T_{X/S}) \times_{\pi_*(T_{X/S \rightarrow Y/S})} T_{Y/S} = \{(\delta, \partial) \in \pi_*(T_{X/S}) \times T_{Y/S} : \pi_*(d\pi)(\delta) = i(\partial)\}.$$

Definition 1.2.1. The sub-sheaf $T_{Y/S}^\pi \subset T_{Y/S}$ of *liftable derivations* on Y is the image of the projection on the second factor

$$T_{Y/S}^\pi = \text{Im}(\pi_*(T_{X/S}) \times_{\pi_*(T_{X/S \rightarrow Y/S})} T_{Y/S} \rightarrow T_{Y/S}).$$

The *discriminant module* $\mathcal{E}_{X/Y}$ is the cokernel of the inclusion morphism $T_{Y/S}^\pi \hookrightarrow T_{Y/S}$ and $D_\pi^{\text{disc}} = \text{supp } \mathcal{E}_{X/Y}$ the *weak discriminant set* of π . Say that π is *weakly submersive* if $\mathcal{E}_{X/Y} = 0$. There are adjoint homomorphisms

$$\begin{aligned} \phi_0 & : \pi^{-1}(\mathcal{E}_{X/Y}) \rightarrow \mathcal{C}_{X/Y}, \\ \phi & : \mathcal{E}_{X/Y} \rightarrow \pi_*(\mathcal{C}_{X/Y}). \end{aligned}$$

Proposition 1.2.2. (1) If $T_{X/Y} = 0$ and ψ_0 (see 1.2) is injective, then ϕ_0 is injective.

(2) If $R^1\pi_*(T_{X/Y}) = 0$ and the composed morphism

$$T_{Y/S} \rightarrow \pi_*\pi^{-1}(T_{Y/S}) \xrightarrow{\pi_*(\psi_0)} \pi_*(T_{X/S \rightarrow Y/S})$$

is injective, then ϕ is injective.

(3) If $\pi_*(T_{X/Y}) = R^1\pi_*(T_{X/Y}) = 0$ (e.g. $T_{X/Y} = 0$) and ψ (see 1.3) is an isomorphism (Prop. 1.1.2), hence $\pi_*(\psi)$ is an isomorphism, then $T_{Y/S}^\pi = \pi_*(T_{X/S})$, $\mathcal{E}_{X/Y} = \pi_*(\mathcal{C}_{X/Y})$, and in particular $D_\pi^{disc} = \pi(C_\pi)$.

If either of the conditions in (1) or (2) holds and π is submersive, then π is weakly submersive.

Assuming the conditions in (1), which is the situation we shall mostly deal with, then in general when ψ is not an isomorphism we have a strict inclusion $D_\pi^{disc} \subset \pi(C_\pi)$, so the notion of submersive morphism is stronger than that of weakly submersive morphism; it is in fact straightforward to see that these notions coincide (assuming (1)) if and only if ψ is an isomorphism. In Example 1.2.3 below the morphism π is weakly submersive but not submersive; the fact that this normalisation morphism is weakly submersive is not coincidental (Th. 2.3.1).

To describe the sheaf $T_{Y/S}^\pi$ we shall use the critical set C_π and the closure D_π of $\pi(C_\pi)$, rather than the weak discriminant set D_π^{disc} (in case $D_\pi^{disc} \neq \pi(C_\pi)$).

Proof. (1): Since $T_{X/Y} = 0$ and π^{-1} is exact we have a commutative diagram of $\pi^{-1}(\mathcal{O}_Y)$ -modules:

$$\begin{array}{ccccccc} 0 & \longrightarrow & T_{X/S} & \longrightarrow & T_{X/S \rightarrow Y/S} & \longrightarrow & \mathcal{C}_{X/Y} \longrightarrow 0 \\ & & \uparrow & & \uparrow \psi_0 & & \uparrow \phi_0 \\ 0 & \longrightarrow & \pi^{-1}(T_{Y/S}^\pi) & \longrightarrow & \pi^{-1}(T_{Y/S}) & \longrightarrow & \pi^{-1}(\mathcal{E}_{X/Y}) \longrightarrow 0 \end{array}$$

Clearly, if ψ_0 is injective, then ϕ_0 is injective.

(2): Letting $\overline{T}_{X/S}$ be the image of $d\pi$ we have a corresponding commutative diagram of \mathcal{O}_Y -modules

$$\begin{array}{ccccccc} 0 & \longrightarrow & \pi_*(\overline{T}_{X/S}) & \longrightarrow & \pi_*(T_{X/S \rightarrow Y/S}) & \longrightarrow & \pi_*(\mathcal{C}_{X/Y}) \longrightarrow 0 \\ & & \uparrow & & \uparrow \pi_*(\psi_0) & & \uparrow \phi \\ 0 & \longrightarrow & T_{Y/S}^\pi & \longrightarrow & T_{Y/S} & \longrightarrow & \mathcal{E}_{X/Y} \longrightarrow 0 \end{array}$$

Here the map from $T_{Y/S}$ to $\overline{T}_{X/S}$ is the composition $T_{Y/S} \rightarrow \pi_*(T_{X/S}) \rightarrow \pi_*(\overline{T}_{X/S})$ where by assumption the latter map is surjective. By the definition of $T_{Y/S}^\pi$ the assertion now follows.

(3): The assumption implies $T_{X/S} = \overline{T}_{X/S}$, and therefore the map ϕ is injective; that ψ is surjective implies that ϕ is surjective; therefore ϕ is an isomorphism.

The remaining assertion is evident, since if ϕ or ϕ_0 are injective, it follows that $\text{supp } \mathcal{E}_{X/Y} \subset \text{supp } \mathcal{C}_{X/Y}$. \square

Let $k \rightarrow A \xrightarrow{\pi} R$ be homomorphisms of rings, and $T_{A/k} = \text{Hom}_A(\Omega_{A/k}, A)$ and $T_{R/k}$ be the Lie algebroid of k -linear derivations of A and R , respectively. The tangent morphism $d\pi : T_{R/k} \rightarrow T_{A/k \rightarrow R/k} = \text{Hom}_A(\Omega_{A/k}, R) = \text{Hom}_R(R \otimes_A \Omega_{A/k}, R)$ is given by $d\pi(\partial)(da) = \partial(\pi(a))$, $\partial \in T_{R/k}$ $a \in A$. Let $\psi_0 : T_{A/k} \rightarrow T_{A/k \rightarrow R/k}$ be the natural morphism, which is injective if π is injective. The liftable derivations are $T_{A/k}^\pi = \text{Im}(d\pi) \cap \psi(T_{A/k})$.

Example 1.2.3. We continue with Example 1.1.1, (i). We have $T_{A/k \rightarrow R/k} = R_t^1 \partial_t$ and $\mathcal{C}_{R/A} = T_{A/k \rightarrow R/k} / T_{R/k} = R/\mathfrak{m}_R = k$, but $T_{A/k}^\pi = T_{A/k}$, so $\mathcal{E}_{R/A} = 0$.

Liftable derivations may be included in $\mathfrak{m}_R^N T_R$ for any high N . We give a random example that N can be high (in this case $N = 9$).

Example 1.2.4. Let A be a \mathbf{Q} -algebra with normalisation $R = \mathbf{Q}[[t]]$, $A \neq R$. Then $T_R^{\text{cont}} = T_{R/\mathbf{Q}}^{\text{cont}} = R\partial_t$ (continuous derivations; alternatively, work instead with the localisation $R = \mathbf{Q}[t]_0$). There exists a smallest integer c such that $(t^c) \in A$. Put $R_c = R/(t^c)$ and $A_c = A/(t^c)$, so $A_c \subset R_c$ is an inclusion of finite-dimensional \mathbf{Q} -algebras. A derivation of R that preserves A is of the form $\nabla = ft\partial_t$, $f \in R$, so ∇ also induces a derivation of A_c and R_c . If $\bar{a}_1, \dots, \bar{a}_l$ are generators of A_c , where $\bar{a}_i \in R_c$ is the projection of $a_i \in R$, the condition on \bar{f} is $\bar{f}\bar{a}'_i \in A_c$, $i = 1, 2, 3$, where $\bar{a}'_i = \overline{t\partial_t(a_i)}$. These equations may have only the trivial solution $f \in (t^c)$. A concrete example is $A = \mathbf{Q}[[a_1, a_2, a_3, t^9, t^{10}, \dots]]$, $a_1 = t^4 + t^7$, $a_2 = t^5 + t^8$, $a_3 = t^6$. We indicate that $T_A^\pi \subset \mathfrak{m}_R^9 T_R$. The ramification index is 4 and $c = 9$, so working in R_9 we see that $1, \bar{a}_1, \bar{a}_2, \bar{a}_3$ forms a \mathbf{Q} -basis for the 4-dimensional subspace A_9 ($\bar{a}_i \cdot \bar{a}_j = 0$, $i \neq j$) of the 9-dimensional space R_9 . To solve $\bar{f}\bar{a}'_i \in A_9$ it suffices to make the ansatz $\bar{f} = c_0 + c_1t + c_2t^2 + c_3t^3 + c_4t^4 \in R_9$, $c_i \in \mathbf{Q}$, so we have the equations $(c_0 + c_1t + c_2t^2 + c_3t^3 + c_4t^4)a'_i \equiv \alpha_{i1}a_1 + \alpha_{i2}a_2 + \alpha_{i3}a_3 \pmod{(t^9)}$, $i = 1, 2, 3$. Identifying coordinates in a basis of the 9-dimensional space R_9 gives 5 equations for each i , so there are 15 equations. There are 5 parameters in \bar{f} and the α_{ij} give another 9 parameters, giving 14 parameters. A straightforward computation shows that the only solution is $\bar{f} = 0$. Note that the vectors $(c_0 + c_1t + \dots + c_4t^4)a'_i$ span a 5-dimensional subspace while the vectors $\alpha_{i1}a_1 + \alpha_{i2}a_2 + \alpha_{i3}a_3$ span a 3-dimensional subspace of R_9 so we should expect that their intersection is 0.

1.3. Logarithmic derivations. Let $k \rightarrow A \xrightarrow{\pi} R$ be inclusions of rings, where we regard A and R as algebras over k , e.g. $k = \mathbf{Z}$, the ring of integers, and R is a discrete valuation ring. Assume that the extension of fraction fields $K(R)/K(A)$ is separably algebraic, and therefore that the A -module of derivations $T_{A/k}$ can be regarded as an A -Lie sub-algebroid of $T_{K(R)/k}$, $T_{A/k} \subset T_{K(R)/k}$; put $T_{A/k}^\pi = T_{A/k} \cap T_{R/k}$.

Definition 1.3.1. Denote by $T_{A/k}(\log^\pi \mathfrak{m}_A)$ the A -submodule of $T_{A/k}$ of derivations ∂ such that $\partial(\phi) \in R\phi$ when $\phi \in \mathfrak{m}_A$, and denote by $T_{A/k}(\mathfrak{m}_A)$ the A -module of derivations such that $\partial(\mathfrak{m}_A) \subset \mathfrak{m}_A$.

The Lie algebroid $T_{A/k}(\log^\pi \mathfrak{m}_A)$ ($= T_{A/k}(\log^\pi \{x_1, \dots, x_r\})$, for a basis $\{x_1, \dots, x_r\}$ of \mathfrak{m}_A) is a sub-Lie algebroid of the Lie algebroid $T_{A/k}(\mathfrak{m}_A)$; elements in the former module are π -logarithmic and elements in the latter are logarithmic derivations.

Lemma 1.3.2. Let $L \subset K(A)$ be a subset. Denote by $T_{A/k}(\log^\pi L)$ the A -submodule of $T_{A/k}$ of derivations ∂ such that $\partial(\phi) \in R\phi$ when $\phi \in L$.

$$T_{A/k}(\log^\pi L) \subset T_{A/k}(\log^\pi \mathfrak{m}_A) \subset T_{A/k}(\mathfrak{m}_A)$$

where the latter is an inclusion of Lie algebroids. If L is either a non-zero A -submodule of $K(A)$ or contains a basis of \mathfrak{m}_A , then $T_{A/k}(\log^\pi L) = T_{A/k}(\log^\pi \mathfrak{m}_A)$.

Note: If P is an \mathfrak{m}_A -primary ideal the Lie algebroids $T_{A/k}(P)$ and $T_{A/k}(\mathfrak{m}_A)$ are in general not included in one another, but if $\text{Char } A = 0$, then $T_{A/k}(P) \subset T_{A/k}(\mathfrak{m}_A)$ (see e.g. [19]). Note also that for general subsets $L \subset K(A)$ we have that $T_{A/k}(L) = \{\partial \in T_{A/k} \mid \partial(\phi) \in L \text{ when } \phi \in L\}$ is a Lie subalgebroid of $T_{A/k}$, but the A -module $T_{A/k}(\log^\pi L)$ need not be a Lie algebra. Moreover, in general $T_{A/k}(\log^\pi L) \not\subset T_{A/k}(L)$, also when L is an ideal of A .

The global version of the above definition is as follows. Let $\pi : X \rightarrow Y$ be a dominant generically separably algebraic morphism of schemes, where X is a Krull scheme, i.e. locally the spectrum of a Krull ring. A Krull ring is an integral domain formed as an intersection $A = \bigcap_\lambda R_\lambda$ of discrete valuation rings R_λ in its fraction field $K(A)$ such that every non-zero element in $K(A)$ is invertible in all but finitely

many R_λ^1 ; a locally Noetherian scheme is Krull if and only if it is integral and integrally closed in its fraction field. Polynomial rings in infinitely many variables over a field are non-Noetherian Krull rings. See [6, Ch VII, §1.3; 9; 24, Ch V] for a treatment of Krull rings.

For a closed subset $D \subset Y$ the sheaf of *logarithmic derivations along D* is $T_{Y/S}(I_D) = T_{Y/S}(\log D) = \{\partial \in T_{Y/S} \mid \partial(I_D) \subset I_D\}$, where I_D is the (reduced) ideal of D . Let $C = \cup C_i$ be a union of irreducible closed subsets of pure codimension 1 in X . Let \mathcal{O}_{X,c_i} be the local ring at the generic point c_i of C_i , and ν_i the associated normalised discrete valuation. The pre-sheaf of π -*logarithmic derivations* along C is

$$T_{Y/S}(\log^\pi C) = \{\partial \in T_{Y/S} \mid \nu_{c_i}(\pi_{c_i}(\partial(f))) \geq \nu_{c_i}(\pi_{c_i}(f)) \\ \text{when } f \in \mathcal{O}_{Y,\pi(c_i)}, i = 1, \dots, r\},$$

where π_c denotes the homomorphism of local rings $\mathcal{O}_{Y,\pi(c)} \rightarrow \mathcal{O}_{X,c}$. It is easy to see that $T_{Y/S}(\log^\pi C)$ actually is a sheaf, and to see that a section ∂ of $T_{Y/S}$ is π -logarithmic it suffices to check that its germ $\partial_{\pi(c_i)} \in T_{\mathcal{O}_{Y,\pi(c_i)}/\mathcal{O}_{S,s_i}}(\log^\pi \mathfrak{m}_{\pi(c_i)})$, where $s_i \in S$ is the image of $c_i \in Y$. Put $D = \pi(C)$ and let I_D be *any* defining ideal of the closure of D , and put $T_{Y/S}(\log^\pi I_D) = \{\partial \in T_{Y/S} \mid \nu_{c_i}(\partial(f)) \geq \nu_{c_i}(f) \text{ for all } \phi \in I, i = 1, \dots, r\}$. The notation is incomplete since this sheaf also depends on the choice of C (different C can give the same D) but we choose not to burden the notation; if $C = \pi^{-1}(D)$ there is no ambiguity and in practice C will always be the critical set C_π of the morphism π . To determine if $\partial \in T_{Y/S}$ belongs to $T_{Y/S}(\log^\pi I_D)$ it suffices to check the condition for a set of local generators of I_D .

Lemma 1.3.3. (1) *The sheaf $T_{Y/S}(\log^\pi I_D)$ is independent of choice of defining ideal I_D of the closure of D ; more precisely*

$$T_{Y/S}(\log^\pi I_D) = T_{Y/S}(\log^\pi C).$$

(2) *If I_D is a defining ideal of D , then*

$$T_{Y/S}(\log^\pi C) \subset T_{Y/S}(\sqrt{I_D});$$

in particular $T_{Y/S}(\log^\pi C) \subset T_{Y/S}(\log D)$.

In view of (1) it is reasonable to write

$$T_{Y/S}(\log^\pi D) = T_{Y/S}(\log^\pi I_D) = T_{Y/S}(\log^\pi C),$$

where I_D is an arbitrary defining ideal of the closure of D .

Proof. (1): Clearly, $T_{Y/S}(\log^\pi C) \subset T_{Y/S}(\log^\pi I_D)$, so assuming ∂ is a section of $T_{Y/S}(\log^\pi I_D)$, regular at $\pi(c_i)$, and $f \in \mathcal{O}_{Y,\pi(c_i)}$, we have to check that $\nu_{c_i}(\partial(f)) \geq \nu_{c_i}(f)$. If $f \notin \mathfrak{m}_{\pi(c_i)}$ we have $\nu_{c_i}(f) = 0$ and the assertion is obvious. If $f \in \mathfrak{m}_{\pi(c_i)}$, then $f^n \in I_{\pi(c_i)}$ for sufficiently high n , hence $\nu_{c_i}(\partial(f^n)) \geq \nu_{c_i}(f^n)$, i.e. $\nu_{c_i}(n\partial(f)f^{n-1}) = \nu_{c_i}(\partial(f)) + (n-1)\nu_{c_i}(f) = n\nu_{c_i}(\partial(f)) \geq n\nu_{c_i}(f)$, so $\nu_{c_i}(\partial(f)) \geq \nu_{c_i}(f)$.

(2): In the light of (1) it suffices to prove that a section ∂ of $T_{Y/S}(\log^\pi I_D)$ preserves the minimal associated primes of I_D for any defining ideal of D . A minimal associated prime of I_D is of the form $\pi(c_i)$ for some generic point c_i of C . Therefore $\partial \in T_{Y/S,\pi(c_i)}(\log^\pi \mathfrak{m}_{\pi(c_i)}) \subset T_{Y/S,\pi(c_i)}(\mathfrak{m}_{\pi(c_i)})$ (Lem. 1.3.2). \square

¹We will need only the first condition for our main results.

1.4. Quasi-finite and algebraic morphisms.

Proposition 1.4.1. *Let $\pi : A \rightarrow R$ be an injective homomorphism of local integral rings, where A is Noetherian. Assume that at least one of the following conditions hold:*

- (i) *A is complete.*
- (ii) *the A -algebra R is of finite type.*
- (iii) *R is integrally closed in $K(R)$, $K(R)$ is finite over $K(A)$, and the integral closure of A in $K(A)$ is finite.*

Then the following are equivalent:

- (1) $\dim_{k_A} k_A \otimes_A R < \infty$ (*R is quasi-finite over A*).
- (2) *R is finite over A .*

See [23] for a discussion of the problem with the property that the normalisation be finite over A . It suffices that the completion of A be reduced, and if R is a discrete valuation ring this is also necessary.

Remark 1.4.2. Recall the situation when A is a discrete valuation ring and $K(R)/K(A)$ is finite. Then k_R/k_A is finite, and the ramification index of $A \rightarrow R$ is the integer e such that $R\mathfrak{m}_A = \mathfrak{m}_R^e$; it is also the index of the value group of A in the value group of R . If A moreover is complete, then $[K(R) : K(A)] = e[k_R : k_A]$.

Proof. We have only to prove (1) \Rightarrow (2). If (i) holds, see [12, Ch. 0, Cor. 7.4.3]. If (ii) holds this is Zariski's main theorem, see [15, Ch. III, Cor. 4.4.6]. If (iii) holds it follows that the integral closure A^* of A in $K(R)$ is finite over A [6, Ch V, §1.6, Prop. 18] and since R is integrally closed, $A^* \subset R$. Now R is a union of local quasi-finite A^* -algebras F of finite type; $K(A^*) = K(R)$ since $K(R)/K(A)$ is algebraic; therefore $K(A^*) = K(F)$, so by Zariski's main theorem $F = A^*$ [15, Ch. III, Cor. 4.4.8]; hence $R = A^*$. \square

Proposition 1.4.3. *Let $\pi : A \rightarrow R$ be a local injective homomorphism of Noetherian integral rings. Consider the conditions:*

- (1) *$K(R)/K(A)$ is algebraic.*
- (2) *k_R/k_A is algebraic.*

Then:

- (a) *If (1) and (2) holds, and A is universally catenary (e.g. formally equidimensional), then $\dim A = \dim R$.*
- (b) *If R is a valuation ring and $A = K(A) \cap R$, then (1) \Rightarrow (2).*
- (c) *If $F\mathfrak{m}_A$ is \mathfrak{m}_F -primary for each local intermediate A -algebra (F, \mathfrak{m}_F) of finite type, $A \subset F \subset R$, then (2) \Rightarrow (1).*

Assuming (2) the assumption in (c) is the same as F/A quasi-finite. Note that if R/A is of finite type and F is a local A -algebra as in (c), then if R/A is quasi-finite it follows that F/A is quasi-finite. To see this, apply $k_A \otimes_A \cdot$ to the exact sequence of A -modules $0 \rightarrow F \rightarrow R \rightarrow F/R \rightarrow 0$, noting that $\dim_{k_A} \text{Tor}_1^A(k_A, R/F) < \infty$ when R is of finite type, and by assumption $\dim_{k_A} k_A \otimes_A R < \infty$. If R/A is not of finite type this does not follow (example: $A = k[x]_0 \subset R = k[[x]]$).

Proof. (a) An algebraic field extension $K(R)/K(A)$ is a union of algebraic extensions $K(F)/K(A)$ where $A \subset F \subset R$ and F is of finite type over A . Since R is Noetherian $\dim R = \sup_{A \subset F \subset R} \dim F$; hence it suffices to see that $\dim F = \dim A$. Since F/A is of finite type we conclude from Ratliff's dimension equality (see [23, Th. 15.6]), since A is Noetherian and universally catenary, that (1 – 2) implies $\dim A = \dim R$.

- (b) See [6, Ch VI, §8, Prop. 1].

(c) By assumption R is a union of local A -algebras F of finite type such that $F\mathfrak{m}_A$ is \mathfrak{m}_F -primary, hence by (2) $F \otimes_A k_A$ is finite over k_A ; hence by Zariski's main theorem (Prop. 1.4.1) F is finite over A . Since $K(R)$ is a union of the fields $K(F)$ it follows that $K(R)$ is algebraic over $K(A)$. \square

1.5. Ramification. We collect some facts about ramifications of injective homomorphisms of local rings $k \rightarrow A \rightarrow R$. The cotangent mapping is denoted

$$\phi : k_R \otimes_{k_A} \mathfrak{m}_A / \mathfrak{m}_A^2 \rightarrow \mathfrak{m}_R / \mathfrak{m}_R^2.$$

A field extension $k \subset l$ has a finite differential basis if $\dim_l \Omega_{l/k} < \infty$, which is equivalent to $\text{tr. deg } l/k < \infty$ when $\text{Char } k = 0$ and equivalent to l/k having a finite p -basis when $\text{Char } k = p$ (see [23, § 26]). We say that an extension $k \subset l$, where k is a subring of the field l , has a finite differential basis if l has a finite differential basis over the fraction field of k .

Proposition 1.5.1. *Let (A, \mathfrak{m}_A, k_A) be a local Noetherian subring of a discrete valuation ring (R, \mathfrak{m}_R, k_R) , where $\mathfrak{m}_A \neq 0$. Let $k \subset A$ be a sub-ring such that the extensions k_R/k and k_A/k are separable with finite differential bases. Consider the following conditions:*

- (1) *The map ϕ is injective.*
- (2) *The ring (A, \mathfrak{m}_A, k_A) is a discrete valuation ring and $T_{A/k} = T_{A/k}^\pi$.*
- (3) *The tangent morphism $d\pi : T_{R/k} \rightarrow T_{A/k \rightarrow R/k}$ is surjective.*
- (4) *The map ϕ is surjective.*

Then (1) \Leftrightarrow (2), (1 – 2) \Rightarrow (3) and (1 – 2) \Rightarrow (4). If A/k is smooth, then (3) \Rightarrow (1 – 2). If A is a discrete valuation ring, then (4) \Rightarrow (1 – 2).

Proof. (1) \Leftrightarrow (2): Since A is a Noetherian integral domain, we have by (1) $1 \leq \dim A \leq \dim_{k_A} \mathfrak{m}_A / \mathfrak{m}_A^2$. Hence (1) implies that (A, \mathfrak{m}_A, k_A) is a discrete valuation ring, thus in either case we have an inclusion of discrete valuation rings $A \subset R$. Since $\dim_{k_R} \Omega_{k_R/k} < \infty$ and $\dim_{k_A} \Omega_{k_A/k} < \infty$ it follows from the second fundamental exact sequence of differentials that $\Omega_{R/k}$ and $\Omega_{A/k}$ are free of finite rank. Therefore $T_{R/k}$ and $T_{A/k \rightarrow R/k} = R \otimes_A T_{A/k}$ (see proof of Proposition 1.1.2) are free of finite rank over R . It follows by Nakayama's lemma that the map $d\pi : T_{R/k} \rightarrow R \otimes T_{A/k}$ is surjective if and only if the tangent map $k_R \otimes_R T_{R/k} \rightarrow k_R \otimes_R T_{A/k}$ is surjective, i.e. the map $(\mathfrak{m}_R / \mathfrak{m}_R^2)^* \oplus T_{k_R/k} \rightarrow k_R \otimes_k (\mathfrak{m}_A / \mathfrak{m}_A^2)^* \oplus k_R \otimes_{k_A} T_{k_A/k}$ is surjective. Since k_R/k_A is separable, the map $T_{k_R/k} \rightarrow T_{k_A/k \rightarrow k_R/k} = k_R \otimes_{k_A} T_{k_A/k}$ is surjective; hence $d\pi$ is surjective if and only if the map $(\mathfrak{m}_R / \mathfrak{m}_R^2)^* \rightarrow k_R \otimes_{k_A} (\mathfrak{m}_A / \mathfrak{m}_A^2)^* = \text{Hom}_{k_A}(\mathfrak{m}_A / \mathfrak{m}_A^2, k_R)$ is surjective and this map is surjective if and only if its image contains the image of the natural map $\text{Hom}_{k_A}(\mathfrak{m}_A / \mathfrak{m}_A, k_A) \rightarrow \text{Hom}_{k_A}(\mathfrak{m}_A / \mathfrak{m}_A, k_R)$, i.e. $d\pi$ is surjective if and only if $T_{A/k} \subset d\pi(T_{R/k})$. Since (1) is equivalent to $(\mathfrak{m}_R / \mathfrak{m}_R^2)^* \rightarrow k_R \otimes_{k_A} (\mathfrak{m}_A / \mathfrak{m}_A^2)^*$ being surjective, it follows that (1) \Leftrightarrow (2).

(1 – 2) \Rightarrow (3): Since $\Omega_{A/k}$ is (free) of finite type the map $d\pi$ is surjective if and only if the map $(\mathfrak{m}_R / \mathfrak{m}_R^2)^* \oplus T_{k_R/k} \rightarrow k_R \otimes_{k_A} (\mathfrak{m}_A / \mathfrak{m}_A^2)^* \oplus k_R \otimes_{k_A} T_{k_A/k}$ is surjective. The assertion is then clear from the above argument, as is the assertion (3) \Rightarrow (1 – 2) when A/k is smooth for then $\Omega_{A/k}$ is free of finite rank since $\dim_{k_A} \Omega_{k_A/k} < \infty$.

(1 – 2) \Rightarrow (4): If ϕ is injective it is also surjective since $\dim \mathfrak{m}_R / \mathfrak{m}_R^2 = 1$. If A and R are discrete valuation rings then ϕ is injective if and only if it is surjective. \square

Proposition 1.5.2. *The following are equivalent for a field extension l/k :*

- (1) *l/k is formally étale and algebraic.*
- (2) *l/k is separably algebraic.*
- (3) *$\Omega_{f/k} = 0$ for each intermediate field $l/f/k$ such that f/k is of finite type.*

If l/k is of finite type or l/k is separable, then (3) is equivalent to $\Omega_{l/k} = 0$.

Example 1.5.3. Let k be a field of characteristic $p > 0$ and $l = k(x^{p^{-\infty}}) = k(x, x^{p^{-1}}, x^{p^{-2}}, \dots)$. Then l/k is separable and formally étale, but not algebraic (see [23, Ex. 26.8]); moreover $\Omega_{l/k} = 0$, and there exist finite intermediate fields $l/f/k$ such that $\Omega_{f/k} \neq 0$.

(3) \Rightarrow (2) is proven in [13, Cor. 21.7.4] when l/k is of finite type, and the argument below is not very different, but it may be convenient to have it presented in this way.

Proof. (1) \Leftrightarrow (2): see [23, Thms. 25.3, 26.9]. For the remaining parts let Ω_k and Ω_l denote differentials over the prime field and consider the exact sequence $0 \rightarrow \Gamma \rightarrow l \otimes \Omega_k \xrightarrow{h} \Omega_l \rightarrow \Omega_{l/k} \rightarrow 0$. (2) \Rightarrow (3): Then l/k is formally étale and h is an isomorphism [*loc. cit.*]. (3) \Rightarrow (2): If $k \subset f \subset l$ is an intermediate field such that f/k is of finite type we have $\Omega_{f/k} = 0$ and replacing l by f in the above exact sequence we see that the map h is surjective and by Cartier's equality $\text{rk}_f \Omega_{f/k} = \text{tr. deg}_k f + \text{rk}_f \Gamma$ it follows that $\text{tr. deg}_k f = 0$ and $\text{rk}_f \Gamma = 0$, hence f/k is algebraic (hence finite) and h is injective, implying that f/k is separable [*loc. cit.*, Th. 26.6]. Since l is a union of such subfields f it follows that l/k is separably algebraic. \square

Theorem 1.5.4. Let $(A, \mathfrak{m}_A, k_A) \subset (R, \mathfrak{m}_R, k_R)$ be an inclusion of local Noetherian rings. Consider the conditions:

- (1) $\Omega_{R/A} = 0$.
- (2) k_R/k_A is separably algebraic and $R\mathfrak{m}_A = \mathfrak{m}_R$.
- (3) k_R/k_A is separably algebraic and the map ϕ is surjective.
- (4) k_R/k_A is separably algebraic and the map ϕ is injective.

Then (2) \Leftrightarrow (3) and (2) \Rightarrow (1). If the residue field extension k_R/k_A is finitely generated, then (1) \Rightarrow (2). If A and R are discrete valuation rings, then (2) \Leftrightarrow (3) \Leftrightarrow (4).

Auslander and Buchsbaum [3] initiated the systematic ramification theory of Noetherian rings. They proved (1) \Leftrightarrow (2) under the assumption that $J = \text{Ker}(R \otimes_A R \rightarrow R, r_1 \otimes r_2 \mapsto r_1 r_2)$ is a finitely generated ideal in $R \otimes_A R$, and hence that $\Omega_{R/A}$ is R -finite. In [14, §17, Th. 17.4.1] one finds a different proof of the same assertion assuming R is a, possibly non-Noetherian, finitely presented A -algebra (implying J is finitely generated), but the argument in [*loc. cit.*] for (1) \Rightarrow (2) is fairly involved. In Theorem 1.5.4 J is not assumed to be finitely generated.

Proof. We give the proof only when A is of equal characteristic. When A is of mixed characteristic one replaces the residue fields by quasi-coefficient rings in the diagram below; these coefficient rings are formally smooth over the integers. Thus we assume that A and R contain the prime field of k_A . Letting Ω_A and Ω_R denote differentials over the prime field we have the exact sequence $0 \rightarrow \Gamma \rightarrow R \otimes \Omega_A \xrightarrow{h} \Omega_R \rightarrow \Omega_{R/A} \rightarrow 0$.

(1) \Rightarrow (2): Since k_R and k_A are separable and hence formally smooth over the prime field [23, Th. 26.9] we have split exact sequences $0 \rightarrow \mathfrak{m}_A/\mathfrak{m}_A^2 \rightarrow k_A \otimes_A \Omega_A \rightarrow \Omega_{k_A} \rightarrow 0$ and $0 \rightarrow \mathfrak{m}_R/\mathfrak{m}_R^2 \rightarrow k_R \otimes_R \Omega_R \rightarrow \Omega_{k_R} \rightarrow 0$ and therefore a commutative diagram which is exact also to the left:

$$\begin{array}{ccccccc}
 0 & \longrightarrow & k_R \otimes_{k_A} \mathfrak{m}_A/\mathfrak{m}_A^2 & \longrightarrow & k_R \otimes_A \Omega_A & \longrightarrow & k_R \otimes_{k_A} \Omega_{k_A} \longrightarrow 0 \\
 & & \downarrow \phi & & \downarrow f & & \downarrow g \\
 0 & \longrightarrow & \mathfrak{m}_R/\mathfrak{m}_R^2 & \longrightarrow & k_R \otimes_R \Omega_R & \longrightarrow & \Omega_{k_R} \longrightarrow 0.
 \end{array}$$

By (1) the map f is surjective, hence g is surjective, so $\Omega_{k_R/k_A} = 0$. By Cartier's equality, using k_R/k_A is finitely generated,

$$\mathrm{rk}_{k_R} \Omega_{k_R/k_A} = \mathrm{tr. deg}_{k_A} k_R + \mathrm{rk}_{k_R} \mathrm{Ker} g,$$

it follows that $\mathrm{Ker} g = 0$ and $\mathrm{tr. deg}_{k_A} k_R = 0$, implying that k_R/k_A is separably algebraic [*loc. cit.*, Th. 26.6]. Therefore g is an isomorphism, in particular injective, hence, as f is surjective, ϕ is surjective by the ‘‘serpent lemma’’; by Nakayama's lemma it follows that $R\mathfrak{m}_A = \mathfrak{m}_R$.

(2) \Rightarrow (1): Assuming (2) the ring R is a union of local A -algebras F of finite type, $A \subset F \subset R$, such that k_F/k_A is separably algebraic and $F\mathfrak{m}_A = \mathfrak{m}_F$. It is easy to see that (1) will follow if $\Omega_{F/A} = 0$ for all such F . Replacing R by F in the above commutative diagram, by (2) the map g is an isomorphism and ϕ is surjective, hence f is surjective, hence, since $\Omega_{F/A}$ is of finite type, by Nakayama's lemma $\Omega_{F/A} = 0$.

(2) \Leftrightarrow (3) is evident.

If A and R are discrete valuation rings, then ϕ is an isomorphism if and only if it is injective or surjective, implying (3) \Leftrightarrow (4). \square

The study of a ramification of a morphism $X \rightarrow Y$ can often be reduced to the case when X is the spectrum of a discrete valuation ring. We now decide what a ramified morphism is in this case.

Definition 1.5.5. Let $\pi : A \rightarrow R$ be an injective homomorphism of local Noetherian rings of the type in Proposition 1.5.1. Then π is *ramified* if the two equivalent conditions (1-2) in Proposition 1.5.1 do not hold. A *tamely ramified* morphism π is a ramified morphism such that the induced fraction field extension $K(R)/K(A)$ is separably algebraic, the residue field extension k_R/k_A is separable, and if $x \in \mathfrak{m}_A \setminus \mathfrak{m}_A^2$, then $\nu(\pi(x))$ is co-prime to the characteristic of A , where ν is the normalised valuation of R . A local domain A of unequal characteristic with $\mathrm{Char} k_A = p > 0$, is (absolutely) unramified if $p \notin \mathfrak{m}_A^2$, and tamely ramified if $p \in \mathfrak{m}_A^n \setminus \mathfrak{m}_A^{n+1}$ for some integer $n \geq 2$ such that $p \nmid n$. A morphism of schemes $\pi : X \rightarrow Y$, where X is regular in codimension ≤ 1 , is tamely ramified if for each point x of height 1 in X the map of local rings $\mathcal{O}_{Y,\pi(x)} \rightarrow \mathcal{O}_{X,x}$ is tamely ramified. An integral scheme Y of unequal characteristic is unramified (tamely ramified) if each local ring $\mathcal{O}_{Y,y}$ is unramified (tamely ramified).

When $\mathrm{Char} k_A = 0$ ramified morphism are always tamely ramified. If R/A is not ramified, then we say that R/A is *unramified*, and if a ramified morphism is not tame, then it is *wild*.

Remark 1.5.6. (i) Assuming $K(R)/K(A)$ and k_R/k_A are separably finite, our definition of tameness is not the same as Abhyankar's [1, §10, p. 245] when A is of mixed characteristic.
(ii) Recall that an A -algebra R is formally unramified if and only if $\Omega_{R/A} = 0$. This is not the same as unramified in Definition 1.5.5, essentially because ϕ in Proposition 1.5.1 need not be injective when it is surjective. Assuming R is a discrete valuation ring, by Theorem 1.5.4 and Proposition 1.5.1 it follows that if R/A is unramified, then R/A is formally unramified; conversely, if R/A is a formally unramified extension of discrete valuation rings, and k_R/k_A is finitely generated, then R/A is unramified.

If R/A has wild ramification we may have $T_{A/k}^\pi \subset \mathfrak{m}^N T_{A/k}$ for high N .

Example 1.5.7. Let k be a field of characteristic p and $\pi : A \cong R = k[y] \rightarrow R = k[x]$ be a morphism over k . If $y = x^n$, and n is co-prime to p , then $T_{A/k}^\pi = Ay \frac{d}{dy}$. If

$y = x^p/(1 - x^{p-1})$, then $T_{A/k}^\pi = Ay^2 \frac{d}{dy}$. One can iterate the latter example, letting π be the composition $A \cong R \rightarrow R \rightarrow \cdots \rightarrow R$ (r times). This gives $T_{A/k}^\pi = Ay^{2r} \frac{d}{dy}$.

Lemma 1.5.8. *Let $\phi_0 : k \rightarrow k'$ be a separably algebraic extension of fields.*

- (1) *A homomorphism $\psi : k[[X]] \rightarrow k'[[X']]$ inducing ϕ_0 on residue fields is (X') -étale if and only if it is (X') -unramified if and only if $\psi(X) \in (X') \setminus (X'^2)$.*
- (2) *If $\text{Char } k = p > 0$ then a local homomorphism of complete p -rings $\phi : (W(k), tW(k), k) \rightarrow (W(k'), t'W(k'), k')$, extending ϕ_0 , is (t') -étale if and only if it is (t') -unramified if and only if $\phi(t) \in (t') \setminus (t'^2)$.*

Proof. Let $R(k)$ denote $k[[t]]$ if $\text{Char } k = 0$ and if $\text{Char } k = p > 0$, it denotes $k[[t]]$ or the p -ring $W(k)$. We have a homomorphism $\phi : (R(k), tR(k), k) \rightarrow (R(k), sR(k), k)$, inducing $\phi_0 : k \rightarrow k'$ on residue fields. If $\phi(t) \in (s^2)$ the morphism is not unramified in the (s) -adic topology, so it suffices to see that if $\phi(t) \in (s) \setminus (s^2)$, then ϕ is étale in the (s) -adic topology. Since it is clearly (s) -unramified it remains to see that ϕ is (s) -smooth. Since in either case $R(k')$ is an integral domain containing the valuation ring $R(k)$ it is flat over $R(k)$; the fibre $R(k')/tR(k') = k'$ is separably algebraic over k , hence formally étale; therefore by [13, Th. 19.7.1] $R(k')$ is (s) -étale over $R(k)$. \square

Remark 1.5.9. Assume that k is perfect so we have unique inclusions $W(f) \subset W(g)$ over field extensions $k'/g/f/k$, and therefore an unambiguous definition of $\lim_{\rightarrow k'/f/k} W(f)$; this ring is formally étale over $W(k)$ (see argument e.g. in [32]).

However, the canonical injective homomorphism $\lim_{\rightarrow k'/f/k} W(f) \rightarrow W(\lim_{\rightarrow k'/f/k} f) = W(k')$, where the limits are over finite intermediate fields f , is not surjective when k'/k is not finite, and there exist no intermediate ring $\lim_{\rightarrow k'/f/k} W(f) \subsetneq R \subset W(k')$

such that $\lim_{\rightarrow k'/f/k} W(f) \rightarrow R$ is finite étale; therefore $W(k')/W(k)$ is not an inductive limit of finite étale morphisms. By Néron desingularisation [25] (recall also Popescu's generalisation in [26] and its new proof in [32]), formally smooth morphisms of discrete valuation rings are filtered inductive limits of finite type smooth morphisms; hence formally étale morphisms are inductive limits of (finite) étale morphisms; therefore $W(k')$ is *not* formally étale over $W(k)$ when k'/k is not finite. Actually, letting $R(k)$ be as in the proof and the morphism $R(k) \rightarrow R(k')$ be (s) -unramified, then it is formally étale if and only if it is finite étale, and this happens if and only if k'/k is separably finite.

2. LIFTABLE DERIVATIONS OF SEPARABLY ALGEBRAIC MORPHISMS

2.1. Lifting derivations to discrete valuation rings. This section contains the main step in the description of liftable derivations. Let $\pi : (A, \mathfrak{m}_A, k_A) \rightarrow (R, \mathfrak{m}_R, k_R)$ be an injective homomorphism of local Noetherian algebras over a ring k , where R is a discrete valuation ring, and assume that A and π are tamely ramified (Def. 1.5.5). In particular, derivations of A are uniquely liftable to derivations of the fraction field of R .

Theorem 2.1.1. *Let $\pi : A \rightarrow R$ be a local injective homomorphism of Noetherian rings, where R is a discrete valuation ring and the extension of fraction fields $K(R)/K(A)$ is separably algebraic.*

- (1) *If π is tamely ramified, then*

$$T_{A/k}^\pi \subseteq T_{A/k}(\log^\pi K(A)) \cap T_{R/k}(\mathfrak{m}_R) \cap T_{A/k}(\mathfrak{m}_A).$$

- (2) Assume that π is residually algebraic and tamely ramified, and if A is of unequal characteristic then A is tamely ramified. Then

$$T_{A/k}^\pi = T_{A/k}(\mathfrak{m}_A).$$

- (3) If $K(A) = K(R)$, then

$$T_{A/k}(\log^\pi \mathfrak{m}_A) \subset T_{A/k}(\mathfrak{m}_A) \cap T_{R/k}(\mathfrak{m}_R).$$

- (4) If π is tamely ramified and $\mathfrak{m}_A \subseteq L \subseteq K(A)$, then

$$T_{A/k}(\log^\pi L) = T_{A/k}^\pi \cap T_{A/k}(\mathfrak{m}_A) = T_{A/k}^\pi.$$

If moreover π is residually algebraic and A is tamely ramified (when A is of unequal characteristic), then

$$T_{A/k}^\pi = T_{A/k}(\mathfrak{m}_A) = T_{A/k}(\log^\pi \mathfrak{m}_A).$$

The main part of Theorem 2.1.1 is (4). There is a certain stability built in in this result, so that the liftable derivations do not change if R is modified by a tamely ramified morphism. Let $\pi : A \xrightarrow{\pi_1} R \xrightarrow{\pi_2} B$, where B also is a discrete valuation ring. Assuming that π_1 is residually algebraic and π_2 is tamely ramified we have

$$T_{A/k}^\pi = T_{A/k}(\log^{\pi_1} \mathfrak{m}_A) \subset T_{R/k}(\mathfrak{m}_R) = T_{R/k}^{\pi_2} \subset T_{B/k},$$

and therefore $T_{A/k}^\pi = T_{A/k}^{\pi_1}$.

Lemma 2.1.2. *Let (A, \mathfrak{m}_A, k_A) be a local N  therian domain and ∂ a derivation of A . Let ν be an additive valuation on the fraction field $K(A)$ with a centre in A . There exists an element c in the (ordered) value group of ν such that for $x \in K(A)$*

$$\nu\left(\frac{\partial(x)}{x}\right) \geq c.$$

Proof. Select generators $\{x_1, \dots, x_r\}$ of the maximal ideal \mathfrak{m}_A . Let ∂ also denote its unique extension to a derivation of $K(A)$. Let c be an element in the value group satisfying $\nu(\partial(x_i)/x_i) \geq c$, $i = 1, \dots, r$, where we may assume $c \leq 0$. Then if $x = ab \in K(A)$, where $a, b \in K(A)$, one has $\partial(x)/x = \partial(a)/a + \partial(b)/b$; hence

$$(2.1) \quad \nu\left(\frac{\partial(x)}{x}\right) \geq \min\left\{\nu\left(\frac{\partial(a)}{a}\right), \nu\left(\frac{\partial(b)}{b}\right)\right\}.$$

If $a \in A$ we have $a = x_{i_1} x_{i_2} \cdots x_{i_n} u$ where $x_{i_j} \in \{x_1, x_2, \dots, x_r\}$, and $u \in A \setminus \mathfrak{m}_A$ is a unit. Then $\nu(\partial(u)/u) \geq 0$ and by (2.1), $\nu(\partial(a)/a) \geq c$. Again by (2.1) we get for any $x = a/b \in K(A)$ that $\nu(\partial(x)/x) \geq c$. \square

Proof of Theorem 2.1.1. Put $p = \text{Char } R$. (1): We first prove $T_{A/k}^\pi \subset T_{R/k}(\mathfrak{m}_R)$, so letting ∂ be a derivation of $K(A)$, hence also a derivation of $K(R)$ since π is generically separably algebraic, such that $\partial(R) \subset R$ and $\partial(A) \subset A$, we have to see that $\partial(\mathfrak{m}_R) \subset \mathfrak{m}_R$.

Assume the contrary, that $\partial(\mathfrak{m}_R) \not\subset \mathfrak{m}_R$. Let $s \in \mathfrak{m}_R \setminus \mathfrak{m}_R^2$ be a uniformising parameter, so any element $x \in R$ is of the form $x = us^n$ for a unit u and positive integer n . From the relation

$$(2.2) \quad \partial(x) = \left(\frac{\partial(u)}{u} + n \frac{\partial(s)}{s}\right)x$$

and the assumption it follows that $\partial(s)$ must be a unit. Let ν be the normalised valuation of R , so $\nu(s^n) = n$, $\nu(\partial(s)) = 0$. Let $t \in \mathfrak{m}_A \setminus \mathfrak{m}_A^2$ have smallest possible value $n = \nu(t)$. Since by tameness $p \nmid n$ it follows that $\partial(t) \neq 0$ and $\nu(\partial(t)) = n - 1$, hence $\partial(t) = u$ is a unit in A and $n = 1$; thus $t \in A$ is a uniformising parameter of R . Replacing ∂ by $u^{-1}\partial$ we can assume that $\partial(t) = 1$. Let $\phi : R \rightarrow k_R$ be the projection to the residue field of R . We regard k_A as a subfield of k_R , using the natural inclusion map. Let $\{t, x_1, \dots, x_r\}$ be a subset of \mathfrak{m}_A that induces a basis of

$\mathfrak{m}_A/\mathfrak{m}_A^2$. Since R/A is ramified we have $r \geq 1$, and writing $x_i = u_i t^{n_i} = u_i t^{\nu(x_i)}$, $u_i \in R \setminus \mathfrak{m}_R$, then $\phi(u_i) \notin k_A$. Consider the set

$$S = \{x \in A \mid \phi(\frac{x}{t^{\nu(x)}}) \in k_R \setminus k_A\},$$

which is nonempty since each x_i belongs to S . If $x = ut^n$, for some $u \in R \setminus \mathfrak{m}_R$, and $p \nmid n$, by (2.2), we have $\nu(\partial(x)) = n - 1$ and hence $\phi(\partial(x)/t^{n-1}) = n\phi(u)$. This gives the following relation in k_R for $x \in A$ such that $p \nmid \nu(x)$ (so in particular $\nu(x) \neq 0$)

$$(2.3) \quad \phi(\frac{\partial(x)}{t^{\nu(\partial(x))}}) = \nu(x)\phi(\frac{x}{t^{\nu(x)}}).$$

Put $n = \nu(S)$, the minimal value of the elements in S . Then $n \geq 1$, for if $x \in A$ and $\nu(x) = 0$, then $\phi(x/t^{\nu(x)}) = \phi(x) \in k_A$, so $x \notin S$. Select $x \in S$ such that $\nu(x) = n$. Then clearly $x \in \mathfrak{m}_A \setminus \mathfrak{m}_A^2$, hence $p \nmid \nu(x)$ since R/A is tamely ramified, hence $\nu(\partial(x)) = n - 1$; by (2.3) we get that $\partial(x) \in S$, contradicting the minimality of n . This completes the proof that $\partial(\mathfrak{m}_R) \subset \mathfrak{m}_R$.

Since $T_{A/k} \cap T_{R/k}(\mathfrak{m}_R) = T_{A/k}(\mathfrak{m}_A)$, we get $T_{A/k}^\pi \subset T_{A/k}(\mathfrak{m}_A)$. It remains to see that $T_{A/k}^\pi \subset T_{A/k}(\log^\pi L)$. Let $a \in L$, so $a = vt^m$ where v is a unit in R , and let $\partial \in T_{A/k}^\pi$. We know that $\partial(t) \in Rt$, hence by (2.2)

$$\frac{\partial(a)}{a} = \frac{\partial(v)}{v} + m \frac{\partial(t)}{t} \in R.$$

Therefore $\partial \in T_{A/k}(\log^\pi L)$.

(2): Continue to regard, by generic separable algebraicity, derivations $\partial \in T_{A/k}(\mathfrak{m}_A)$ as derivations of $K(R)$ and in particular as derivations $R \rightarrow K(R)$. We need to prove that $\partial(R) \subset R$. Let ν be the normalised valuation of $K(R)$ whose valuation ring is R . By Lemma 2.1.2 there exists an integer c such that $\nu(\partial(r)) \geq \nu(r) - c$ for $r \in R$, hence $\partial(\mathfrak{m}_R^n) \subset \mathfrak{m}_R^{n-c} \subset K(R)$. Therefore, providing $K(R)$ with the linear topology where $\{\mathfrak{m}_R^n\}$ is family of neighbourhoods of 0, it follows that ∂ is a continuous derivation $R \rightarrow K(R)$; hence it has a unique continuous extension to the completed rings $\partial : \bar{R} \rightarrow \varprojlim_n K(R)/\mathfrak{m}_R^n \cong K(\bar{R})$; hence it has a unique continuous extension to a derivation $K(\bar{R}) \rightarrow K(\bar{R})$. By Artin-Rees' lemma $R \subset \bar{R}$, where \bar{R} also is a discrete valuation ring [6, VI, § 5.3, Prop 5]. Let $r \in R$. If $\partial(r) \notin R$, then $1/\partial(r) \in \mathfrak{m}_R \subset \mathfrak{m}_{\bar{R}}$, so $\partial(r) \notin \bar{R}$. Therefore it suffices to see that $\partial(\bar{R}) \subset \bar{R}$. The map $\pi : A \rightarrow R$ induces a map $\bar{\pi} : \bar{A} \rightarrow \bar{R}$ of completed rings. Since \bar{R} is integral and \bar{A} is 1-dimensional (Lem. 1.4.3) $\text{Ker } \bar{\pi}$ is either $\mathfrak{m}_{\bar{A}}$ or 0; since $A \subset \bar{A}$ and $R \subset \bar{R}$ by Artin-Rees' lemma it follows that $\bar{\pi}(A) \neq 0$ and therefore $\bar{\pi}$ is injective (cf. Remark 2.1.3). Since $T_A(\mathfrak{m}_A) \subset T_{\bar{A}}(\mathfrak{m}_{\bar{A}})$ it should be clear now that one may assume that $A = \bar{A}$ and $R = \bar{R}$.

There are two cases:

(a) A and R are complete of unequal characteristic: Put $p = \text{Char } k_A = \text{Char } k_R$. The rings A and R contain complete p -rings $W(k_A) \subset A$, $W(k_R) \subset R$, and the field extension k_R/k_A lifts to a formally unramified inclusion of p -rings $(W(k_A), tW(k_A), k_A) \subset (W'(k_R), sW'(k_R), k_R)$ taking t to s [23, Thms. 29.1-2], which is hence $\mathfrak{m}_{W'(k)}$ -étale (Lem. 1.5.8). Since all complete p -rings with given residue field are isomorphic [23, Cor. 29.2] we have $W'(k_R) \cong W(k_R)$; we therefore identify $W'(k_R)$ with $W(k_R)$, so $W(k_A) \rightarrow W(k_R) \subset R$ is $\mathfrak{m}_{W(k_R)}$ -étale. Our derivation $\partial : W(k_A) \rightarrow A$ induces a homomorphism of rings

$$\phi(\partial) : W(k_A) \rightarrow \frac{A[X]}{(X^2)} \subset R[X]/(X^2),$$

extending the inclusion $W(k_A) \rightarrow R$, and by $\mathfrak{m}_{W(k_R)}$ -étaleness it lifts uniquely to a continuous homomorphism

$$\phi' : W(k_R) \rightarrow (\varprojlim R/\mathfrak{m}_R^n)[X]/(X^2) = \frac{R[X]}{(X^2)},$$

extending the inclusion $W(k_R) \rightarrow R$; such a homomorphism is of the form $\phi' = \phi(\partial')$ where ∂' is a derivation $W(k_R) \rightarrow R$. This implies

$$\partial(W(k_R)) = \partial'(W(k_R)) \subset R.$$

By assumption the local ring extensions $A/W(k_A)$, R/A are tamely ramified, implying that $R/W(k_R)$ is tamely ramified. The extension $R/W(k_R)$ is described by the relation for a uniformising parameter $x \in \mathfrak{m}_R \setminus \mathfrak{m}_R^2$: there exists an Eisenstein polynomial $f(X) = X^n + a_1X^{n-1} + \dots + a_n \in W(k_R)[X]$, $p|a_i$, $1 \leq i \leq n$, $p^2 \nmid a_n$ (the divisibility takes place in $W(k_R)$), such that $f(x) = 0$; hence $p = ux^n$, $u \in R \setminus \mathfrak{m}_R$; therefore by tameness $p \nmid n$. Applying ∂ gives $f'(x)\partial(x) + f^\partial(x) = 0$, where $f^\partial(x) = x^{n-1}\partial(a_1) + \dots + \partial(a_n)$. Since $\partial(a_i) \in R$ and $\partial(p) = 0$ we have $\partial(a_i) = r_i p = r'_i x^n$, for some $r_i, r'_i \in R$, and it follows that one can write $f^\partial(x) = rx^n$, for some $r \in R$. Since $p \nmid n$ we have $f'(x) = nx^{n-1} + (n-1)a_1x^{n-2} + \dots + a_{n-1} = ux^{n-1}$, where $u \in R \setminus \mathfrak{m}_R$. It follows that $\partial(x) = -f^\partial(x)/f'(x) = -rx/u \in \mathfrak{m}_R \subset R$.

(b) A and R are complete of equal characteristic: Then A has a coefficient field $k_A \subset A$ which injects into R and this image is part of a coefficient field k_R of R , so $k_A \subset k_R \subset R = k_R[[t]]$, where $t \in \mathfrak{m}_R \setminus \mathfrak{m}_R^2$. Since k_R/k_A is separably algebraic there exists to each $\alpha \in k_R$ a polynomial $f \in k_A[X]$ such that $f(\alpha) = 0$, $f'(\alpha) \neq 0$. Applying ∂ gives $f'(\alpha)\partial(\alpha) + f^\partial(\alpha) = 0$, hence $\partial(\alpha) \in k_R \subset R$. This implies

$$\partial(k_R) \subset R.$$

It remains to see that $\partial(t) \in Rt$. Assume the contrary, $\partial(t) \notin (t)$. Choose $x \in A$ such that $R\mathfrak{m}_A = Rx$, and write $x = ut^n$, where u is a unit in R . Since $\partial(k_R) \subset R$ we have $\nu(\partial(u)) \geq \nu(\partial(t))$. Therefore, since n and p are coprime, $\nu(\partial(x)) < \nu(x)$. This contradicts the assumptions $\partial(x) \in \mathfrak{m}_A$ and $\nu(\mathfrak{m}_A) = \nu(x)$.

(3): Let $\partial \in T_{A/k}(\log^\pi \mathfrak{m}_A)$. Then for $a \in A$ we have $\partial(a) = ra \in A$ for some $r \in R$. Any element $z \in K(A)$ can be written a/b with $a, b \in \mathfrak{m}_A$, implying

$$\frac{\partial(z)}{z} = \frac{\partial(a)}{a} - \frac{\partial(b)}{b} \in R.$$

Since $K(R) = K(A)$ it follows $\partial(R) \in R$ and $\partial(\mathfrak{m}_R) \subset \mathfrak{m}_R$; moreover $T_{A/k}^\pi \cap T_{R/k}(\mathfrak{m}_R) \subset T_{A/k}(\mathfrak{m}_A)$. (When R/A is tamely ramified the inclusion is an equality by (1).)

(4): By (1) $T_{A/k}^\pi \subset T_{R/k}(\mathfrak{m}_R) \cap T_{A/k}(\log^\pi \mathfrak{m}_A) \subset T_{A/k}(\mathfrak{m}_A)$, so it remains to see that $T_{A/k}(\log^\pi \mathfrak{m}_A) \subset T_{A/k}^\pi$. Put $R_1 = R \cap K(A)$, which again is a discrete valuation ring and k_R/k_{R_1} is algebraic (Lem. 1.4.3). We have inclusions of local rings

$$A \xrightarrow{\pi_1} R_1 \xrightarrow{i} R,$$

so $\pi = i \circ \pi_1$. That π is tamely ramified implies that π_1 and i are tamely ramified. Let $\partial \in T_{A/k}(\log^\pi \mathfrak{m}_A)$. Then for $a \in A$ we have $\partial(a) = ra \in A$ for some $r \in R$, hence $r \in R \cap K(A) = R_1$, hence $\partial(a) \in R_1 a$; therefore

$$T_{A/k}(\log^\pi \mathfrak{m}_A) \subset T_{A/k}(\log^{\pi_1} \mathfrak{m}_A) \subset T_{A/k}(\mathfrak{m}_A) \cap T_{R_1/k}(\mathfrak{m}_{R_1})$$

where the last inclusion follows from (3). In particular, $T_{A/k}(\log^\pi \mathfrak{m}_A) \subset T_{R_1/k}(\mathfrak{m}_{R_1})$. Since i is tamely ramified and k_R/k_{R_1} is algebraic, by (2) we have $T_{R_1/k}(\mathfrak{m}_{R_1}) \subset T_{R_1/k}^i$; therefore $T_{A/k}(\log^\pi \mathfrak{m}_A) \subset T_{A/k}^\pi$. \square

Remark 2.1.3. If $A \rightarrow R$ is a local injective map of Noetherian rings, it does not follow that the induced map of completed rings $\bar{\pi} : \bar{A} \rightarrow \bar{R}$ is injective (see [10, 17]) but derivations of A extend to derivations of \bar{A} that preserve the kernel of $\bar{\pi}$, since the identity $\pi \circ \partial - \partial \circ \pi = 0$ extends, by continuity, to the identity $\bar{\pi} \circ \partial = \partial \circ \bar{\pi}$.

2.2. Proof of the lifting theorem.

Theorem 2.2.1. *Let $\pi : X/S \rightarrow Y/S$ be a morphism of schemes over a scheme S . Assume that Y and S are Noetherian and X is a Krull scheme. Assume that π is dominant, generically algebraic and tamely ramified (Def. 1.5.5). If Y is of unequal characteristic, assume that Y is absolutely tamely ramified. Let D_π be the discriminant set of π . Then:*

(L₁) *Let J_{D_π} be any defining ideal of the closure of D_π . Then*

$$T_{Y/S}^\pi = T_{Y/S}(\log^\pi J_{D_\pi}).$$

(L₂) *If the residue field extension $k_c/k_{\pi(c)}$ is algebraic for each point $c \in C_\pi$ of height 1, then*

$$T_{Y/S}^\pi = T_{Y/S}(I_{D_\pi}),$$

where I_{D_π} is the (reduced) ideal of the closure of the discriminant set D_π .

(L₃) *Let I_{C_π} be the ideal of the critical locus of π . Then $\pi^{-1}(T_{Y/S}^\pi) \subset T_{X/S}(I_{C_\pi})$, i.e. the liftable vector fields are tangent to the critical locus.*

Proof. Since X/Y is generically separably algebraic any derivation ∂ of $\mathcal{O}_Y(U)$, where U is open in Y , has a unique lift $\bar{\partial}$ to a derivation of the field $\mathcal{O}_{X,\xi}$ over the generic point ξ of X . Since X is Krull it suffices to see that $\bar{\partial}(\mathcal{O}_{X,c}) \subset \mathcal{O}_{X,c}$ when $c \in X$ is a point of height 1 that maps to U , for then $\bar{\partial}(\mathcal{O}_{\pi^{-1}(U)}) \subset \mathcal{O}_{\pi^{-1}(U)}$. If $c \notin C_\pi$, i.e. $d\pi_c$ is surjective, this is evidently the case, so assume $c \in C_\pi$ is a point of height 1 in X . Let c be a point of C_π of height 1. Put $R = \mathcal{O}_{X,c}$, $A = \mathcal{O}_{Y,\pi(c)}$, so we may regard A as a subring of the discrete valuation ring R , with inclusion morphism $\pi : A \rightarrow R$. Put also $k = \mathcal{O}_{S,s}$. Let \mathfrak{m}_A and \mathfrak{m}_R be the maximal ideal of A and R , so $\mathfrak{m}_A \subset \mathfrak{m}_R$. We need to prove that $T_{A/k}^\pi = T_{A/k}(\log^\pi \mathfrak{m}_A)$, and if k_R/k_A is algebraic, $T_{A/k}^\pi = T_{A/k}(\mathfrak{m}_A)$. By Proposition 1.5.1, (1) \Rightarrow (3), the morphism $A \rightarrow R$ is ramified (hence tamely ramified). Therefore (L₁) follows from Theorem 2.1.1, (3), and (L₂) follows from Theorem 2.1.1, (2). The assertion in (L₃) follows from Theorem 2.1.1, (1). \square

2.3. The normalisation morphism. Let $\pi : \bar{X}/S \rightarrow X/S$ be the normalisation of an integral Noetherian scheme.

Theorem 2.3.1. (1) $T_{X/S}(\log^\pi I_D) \subset T_{\bar{X}/S}^\pi$.

(2) *If the normalisation morphism π is tamely ramified, then $T_{\bar{X}/S}^\pi = T_{X/S} \subset T_{\bar{X}/S}(I_C)$ (in particular, π is weakly submersive).*

Proof. To alleviate the notation slightly we consider only the case $S = \text{Spec } \mathbf{Z}$; the general relative case is no different. Let $x \in \bar{X}$, $y = \pi(x)$ and put $A = \mathcal{O}_x$, $B = \mathcal{O}_y$, where y lies over a prime number s . We have an inclusion of integral domains $A \subset B$ with coinciding fraction fields, $K = K(A) = K(B)$, and B is integrally closed in K . We need to prove that a derivation $\partial : A \rightarrow A$ can be lifted to a derivation $\tilde{\partial} : B \rightarrow B$. First, ∂ can be extended uniquely to a derivation $\tilde{\partial} : K \rightarrow K$. By the Mori-Nagata theorem [24, Th. 33.10] the normalisation of a Noetherian integral domain is a Krull ring, hence B is the intersection of the discrete valuation rings R in K containing B . Hence we may regard A as a subring of a discrete valuation ring R and it suffices to see that $\partial(R) \subset R$. Localise A at the

prime $p \in \text{Spec } B$ such that $R = B_p$. Then $A_p \subset R$ and $\partial(A_p) \subset A_p$, so one may assume $A = A_p$. We have then an algebraic extension $(A, \mathfrak{m}_A, k_A) \rightarrow (R, \mathfrak{m}_R, k_R)$ of one-dimensional local rings where R is regular. Let ν be the normalised discrete valuation of R . If $A = R$ there is nothing to prove, hence one may assume that A is not regular (i.e. not normal).

(1): Let $\partial \in T_A(\log^\pi \mathfrak{m}_A)$. The proof that $\partial(R) \subset R$ is similar to the proof of (3) in Theorem 2.1.1. Any element in $K(A) = K(R)$ can be written $z = a/b$ with $a, b \in \mathfrak{m}_A$. Since $\partial(a) \in Ra$ for each $a \in \mathfrak{m}_A$, we get

$$\frac{\partial(z)}{z} = \frac{\partial(a)}{a} - \frac{\partial(b)}{b} \in R.$$

In particular, $\partial(R) \subset R$. This proves $T_A(\log^\pi \mathfrak{m}_A) \subset T_A^\pi$.

(2): Let ∂ be a derivation of A , inducing a derivation of $K(A) = K(R)$, and reasoning as in the proof of Theorem 2.1.1, (2), it induces a derivation $\bar{R} \rightarrow K(\bar{R})$; we have $A \subset R \subset \bar{R}$. Since $\bar{R} \cap K(R) = R$ it suffices to see that $\partial(\bar{R}) \subset \bar{R}$. Let $t \in \mathfrak{m}_{\bar{R}} \setminus \mathfrak{m}_{\bar{R}}^2$ be a uniformising parameter. Let $W(k_R)$ either be a complete p -ring such that \bar{R} is an Eisenstein extension of $W(k_R)$, or a coefficient field in \bar{R} , and $W(k_A)$ either be a complete p -ring or a coefficient field in A such that $W(k_A) \subset W(k_R)$. Since B/A is integral, so the localisation $A_q \subset B_p = R$ is integral, where p is a prime ideal in B and $q = A \cap p$, by assumption the field extension k_R/k_{A_q} is separably algebraic. We can assume that $A = A_q$. Now follow the proof of (2) in Theorem 2.1.1. First we get $\partial(W(k_R)) \subset R$, which is a consequence of the fact that $W(k_R)/W(k_A)$ is $\mathfrak{m}_{W(k_R)}$ -étale. Note that this follows without a priori assuming $\partial(\mathfrak{m}_A) \subset \mathfrak{m}_A$. Secondly, we get $\partial(\bar{R}) \subset \bar{R}$ if $\partial(t) \in \bar{R}$. Finally, by Theorem 2.1.1, (1), if $T_A = T_A^\pi$, it follows that $T_A = T_A(\mathfrak{m}_A) \subset T_R(\mathfrak{m}_R)$.

Assume the contrary, that $\nu(\partial(t)) < 0$, where ν is the normalised valuation of \bar{R} . Let x_1, \dots, x_n be a system of parameters of \mathfrak{m}_A and put $m_i = \nu(x_i)$, where we may assume $m_1 \leq m_2 \leq \dots \leq m_n$. Since $p \nmid m_i$, we get

$$(2.4) \quad \nu(\partial(x_i)) = \nu(x_i) + \nu\left(\frac{\partial(u_i)}{u_i} + m_i \frac{\partial(t)}{t}\right) = m_i + \nu(\partial(t)) - 1,$$

where $x_i = u_i t^{m_i}$ for some unit $u_i \in R$. The last equality follows since $\partial(W(k_R)) \subset R$, implying $\nu(\partial(u)) \geq \nu(\partial(t))$ (since $\nu(\partial(t)) < 0$); this holds both in equal and mixed characteristic. Since $\partial(x_1) \in A$ and $\nu(\partial(t)) < 0$, so $0 \leq m_1 + \nu(\partial(t)) - 1 < m_1$, we get $\nu(\partial(t)) = 1 - m_1$ and $m_1 \geq 2$. To get a contradiction it suffices to see that $m_1 \mid m_i$, $i = 1, 2, \dots, n$, since then $\mathbf{Z}m_1 + \mathbf{Z}m_2 + \dots + \mathbf{Z}m_n$ is a proper ideal of \mathbf{Z} which would be in opposition to $K(A) = K(R)$. Since $m_1 \mid m_1$, assume $m_1 \mid m_i$ when $i = 1, 2, \dots, r-1$ for some integer $r \geq 2$. We get from (2.4)

$$\nu(\partial(x_r)) = m_r - m_1 = \sum_{i=1}^{r-1} a_i m_i,$$

where $a_i \in \mathbf{N}$; the last equality follows since $\nu(\partial(x_r)) < m_r$. Hence $m_1 \mid m_r$. \square

Corollary 2.3.2. *Let (A, \mathfrak{m}_A) be a local 1-dimensional integral Noetherian ring with a tamely ramified normalisation. If there exists a derivation ∂ of A and an element $x \in \mathfrak{m}_A \setminus \mathfrak{m}_A^2$ such that $\partial(x) \notin \mathfrak{m}_A$, then A is regular.*

Proof. From the above proof we get $T_A = T_A(\mathfrak{m}_A)$ if A is 1-dimensional and not normal, implying the assertion. \square

Remark 2.3.3. Corollary 2.3.2 follows from Zariski's lemma when A contains the rational numbers and its completion is reduced (see [23, Corollary to Th. 30.1]). A more instructive argument that $T_{A/k} = T_{A/k}(\mathfrak{m}_A)$ for singular one-dimensional k -algebras A can be achieved in a special case. Assume that A is a local ring such that the Jacobian criterion of regularity holds. Then the Jacobian ideal J

is preserved by $T_{A/k}$, i.e. $T_{A/k} = T_{A/k}(J)$. If A is of characteristic 0 we have $T_{A/k} = T_{A/k}(\sqrt{J})$ (see e.g. [19]), hence if A is singular we have a proper $T_{A/k}$ -invariant radical ideal $\sqrt{J} \subset \mathfrak{m}_A$. Therefore, if A is a singular one-dimensional ring it follows that $T_{A/k} = T_{A/k}(\mathfrak{m}_A)$. In Corollary 2.3.2 no assumption is made on the characteristic or the validity of the Jacobian criterion. However, for wildly ramified normalisations the existence of $\partial \in T_{A/k}$ and $t \in \mathfrak{m}_A$ such that $\partial(t) = 1$ does not imply that A is regular (Ex. 2.3.5).

Remark 2.3.4. What Seidenberg actually proved is this. Let A be an integral domain, containing the rational numbers \mathbf{Q} , \bar{A} its integral closure and A' the ring of quasi-integral elements (so $\bar{A} = A'$ when A is Noetherian), then $T_{A/\mathbf{Q}} \subset T_{A'/\mathbf{Q}}$, and if the ring of formal power series $\bar{A}[[t]]$ is integrally closed, then $T_{A/\mathbf{Q}} \subset T_{\bar{A}/\mathbf{Q}}$. Say that a derivation ∂ of A is integrable if there exists a Hasse-Schmidt derivation of the form $(1, \partial, D_2, \dots)$. Note that the higher derivations D_i are not uniquely determined by ∂ ; see [22]. The idea in Seidenberg's proof is to consider only the integrable derivations, giving rise to automorphisms of $A[[t]]$, and the point is that derivations are always integrable to higher derivations when $\mathbf{Q} \subset A$. On the other hand, if $\mathbf{Q} \not\subset A$ the question of integrability is a delicate matter. The only general condition I am aware of that all derivations of a k -algebra A be integrable is that A/k be smooth [22], but this is of no interest for the lifting problem, at least when k is a perfect field, since then A is normal. Letting $T_{X/S}^{int}$ be the sheaf of integrable derivations, Seidenberg's theorem states that $T_{X/S}^{int} \subset T_{X/S}^\pi$ when π is the normalisation morphism of a Noetherian scheme [30] (also when π is wildly ramified). Assume that X/k is a variety. Matsumura [22] proves that if k is a perfect field, then $T_{X/k}^{int}$ is of the same rank as $T_{X/k}$, and gives an example that over an imperfect field k one may have $T_{X/k}^{int} = 0$ while $T_{X/k} \neq 0$. On the other hand, $T_{X/S}(\log^\pi \mathfrak{m}_A) \subset T_{X/k}^\pi$ (Th. 2.3.1), and this module is always non-zero when $T_{X/k} \neq 0$.

We extend an example by Seidenberg [30] giving many examples of rings with wild normalisations and where there exist derivations that do not lift.

Example 2.3.5. Let R be a discrete valuation ring of characteristic $p > 0$, K its fraction field and ∂ a derivation of R such that $\partial(\mathfrak{m}_R) \not\subset \mathfrak{m}_R$; we also regard ∂ as a derivation of K and let $K^\partial \subset K$ be the subfield of elements r such that $\partial(r) = 0$ (so $R^p \subset K^p \subset K^\partial$). Suppose S is a subset of R such that $\partial(S) \subset S$ and $\partial^n(S) = 0$ for some positive integer n . The set $\mathfrak{m}_R^\partial = K^\partial \cap \mathfrak{m}_R$ is non-trivial if and only if $\text{Char } R = p > 0$, and then $\mathfrak{m}_R^\partial \subset \mathfrak{m}_R^p$. Select an element $c \in \mathfrak{m}_R^\partial$ and an increasing sequence of subsets $B_0 \subset B_1 \subset \dots \subset B_n \subset \mathfrak{m}_R^\partial$ such that $B_i \subset cB_{i+1}$ and let A be the smallest local subring of R that contains the set

$$S_B = SB_0 + \partial(S)B_1 + \dots + \partial^{n-1}(S)B_n.$$

Since clearly $\frac{1}{c}\partial(S_B) \subset S_B$ we get $\frac{1}{c}\partial(A) \subset A$, but $\frac{1}{c}\partial(R) \not\subset R$. If S generates K and $S \cap B_0 \neq \emptyset$, then $K(A) = K$ and the inclusion $\pi : A \rightarrow R$ is the normalisation of A . The derivation $\delta = \frac{1}{c}\partial$ of A does not belong to $T_A(\log^\pi \mathfrak{m}_A)$. Let k be a field of characteristic p , $R = k[t]_0$, $S = \{t^p, t^{p+1}\} \subset k[t]_0$ and $B_i = t^{(i+1)p}$. Then $A = k[t^p, t^{p+1}]_0 \subset R = k[t]_0 \subset k(t)$, which is an extension with ramification index p . This is close to Seidenberg's example. Then $\delta = t^{-p}\partial_t$ is a derivation of A such that $\delta(R) \not\subset R$. Clearly, $\delta(t^{p+1}) \notin Rt^{p+1}$, so $\delta \notin T_A(\log^\pi \mathfrak{m}_A)$. Note also that $\partial_t \in T_A^\pi$, but $\partial_t(\mathfrak{m}_R) \not\subset \mathfrak{m}_R$, showing that (1) in Theorem 2.1.1 need not hold when π is wildly ramified.

All derivations may lift even if the normalisation is wildly ramified.

Example 2.3.6. Put $A = k[t^3, t^7, t^8]_0 \subset R = k[t]_0$ and assume $\text{Char } k = 3$. Here $T_{A/k} = A\partial_t \subset T_{R/k}$.

3. BIRATIONAL MORPHISMS

3.1. Differentially ramified morphisms. Assuming the morphism π is of the type in Theorem 2.2.1 we have $T_{X/Y} = 0$. Therefore, if J is an ideal of \mathcal{O}_X it makes sense to define the sub-sheaf of $T_{X/S}$ of liftable derivations that preserve J by

$$T_{Y/S}^\pi(J) = T_{Y/S}^\pi \cap T_{X/S}(J).$$

For an ideal I of \mathcal{O}_Y , its inverse image ideal is $\tilde{I} = \mathcal{O}_X I$ (with obvious interpretation).

Definition 3.1.1. A dominant morphism of schemes $\pi : X \rightarrow Y$, where X is Krull and Y locally Noetherian, with discriminant locus D , is *uniformly ramified* at a generic point ξ of the critical locus C if the stalk $I_{D, \pi(\xi)}$ of the ideal of D at $\pi(\xi)$ has a basis $I_{D, \pi(\xi)} = (x_1, \dots, x_l)$ satisfying $\nu_\xi(x_i) = \nu_\xi(x_j)$, for each i, j . The morphism π is uniformly ramified if it is uniformly ramified at each generic point of the critical locus C .

Thus the condition is on the base Y but local at each generic point of C . Do not confuse this with another notion of uniform ramification in [11].

Definition 3.1.2. A generically finite morphism π is *differentially ramified* if

$$T_{Y/S}^\pi = T_{Y/S}(I_D),$$

where I_D is the (reduced) ideal of the discriminant D .

Thus for differentially ramified morphisms the liftable derivations can be characterised in terms of the discriminant D only, and do not depend on the particular differentially ramified morphism with such a discriminant. For example, residually algebraic morphisms as in Theorem 2.2.1 are differentially ramified.

Theorem 3.1.3. Let $\pi : X \rightarrow Y$ be a morphism as in Theorem 2.2.1.

- (1) If π is uniformly ramified, then it is differentially ramified.
- (2) Assume that for each generic point ξ of C , there exists a basis $I_{D, \pi(\xi)} = (x_1, \dots, x_r)$ and derivations $\partial_1, \dots, \partial_r \in T_{Y/S, \pi(\xi)}$ such that $\det \partial_j(x_i)$ is invertible. Then π is uniformly ramified if and only if it is differentially ramified.

For example, by (1), if D_π is a hypersurface, i.e. I_{D_π} is locally principal, then morphisms of the considered type are differentially ramified.

Proof. (1): If π is uniformly ramified a derivation $\partial \in T_{Y/S}(I_D)$ satisfies, for each generic point ξ of C , $\nu_\xi(\partial(x_i)) \geq \nu_\xi(x_i)$ for some basis (x_1, \dots, x_l) of $I_{D, \pi(\xi)}$. The assertion therefore follows from Theorem 2.2.1.

(2): By (1) we have only to prove that π is uniformly ramified when $T_{Y/S}^\pi = T_{Y/S}(I_D)$. Let c_{ij} be the inverse matrix of $(\partial_i(x_j))$. Setting $\partial'_i = \sum c_{ij} \partial_j$ we have $\partial'_i(x_j) = \delta_{ij}$, so that we can assume $\partial_i(x_j) = \delta_{ij}$. By assumption $T_{Y/S}(I_D) \subset T_{Y/S}(\log^\pi I_D)$ (Th. 2.2.1); hence $\nu_\xi(x_i) = \nu_\xi(x_i \partial_j(x_j)) \geq \nu_\xi(x_j)$. By symmetry the opposite inequality also holds. \square

3.2. Blow-ups. A projective birational morphism of integral Noetherian schemes $\pi : \tilde{X} \rightarrow X$ can be constructed as a blow-up of a coherent fractional ideal on X , and if X and Y are projective over a field one may even construct π as a blow-up of an ideal [15, Prop. 2.3.5, Cor. 2.3.7]. Let X be an integral scheme with sheaf of rational functions R_X , let I be a coherent \mathcal{O}_X -submodule of R_X , and let $\pi : Bl_I(X) \rightarrow X$ be the blow-up of X along I . We will in this section try to describe the liftable derivations $T_{X/S}^\pi$, without using valuations along the critical locus of π ; more precisely, we look for a (fractional) ideal J such that $T_{X/S}^\pi = T_{X/S}(J)$. Clearly, if such an ideal exists, then $D_\pi = \{x \in X \mid J_x \neq \mathcal{O}_{X,x}\}$, the discriminant locus of π , so J and I would have the same radical.

Let $p : \overline{Bl}_I(X) \rightarrow Bl_I(X)$ be the normalisation morphism of the blow-up of I . The integral closure of I is the sheaf \bar{I} whose sections over an open set U is

$$\bar{I}(U) = \bigcap_{x \in U, \text{ht}(x) \leq 1} \mathcal{O}_{\overline{Bl}_I(X), x} I_x.$$

Lemma 3.2.1. (*see [33]*)

$$\begin{aligned} \bar{I}(U) &= \{f \in R_X(U) \mid \text{there exists an equation } f^n + b_1 f^{n-1} + \cdots + b_n = 0, \\ &\quad n > 0, b_i \in I^i(U)\} \\ &= (\pi \circ p)_*(I \mathcal{O}_{\overline{Bl}_I(X)}). \end{aligned}$$

Lemma 3.2.1 implies that \bar{I} is coherent since $\pi \circ p$ is proper. Clearly, $I \subset \bar{I}$, and if X is normal and I an ideal of \mathcal{O}_X then $\bar{I} \subset \mathcal{O}_X$. See [16] for relations between $\overline{Bl}_I(X)$ and $Bl_{I^n}(X)$ for high n .

Define $[I^{n+1} : I^n] = \{a \in R_X \mid aI^n \subset I^{n+1}\}$; this is a coherent fractional ideal. Then for sufficiently high n we have $[I^{n+2} : I^{n+1}] = [I^{n+1} : I^n]$. Assuming that I be coherent, the Ratliff-Rush fractional ideal \hat{I} associated to I is the common fractional ideal $[I^{n+1} : I^n]$ for high n , and we have $\hat{I} \subset \bar{I}$. One has $I^n = (\hat{I})^n$ for high n [loc. cit.]. These assertions are well-known when I is a coherent ideal [27] and they immediately generalise to fractional ideals since a coherent fractional ideal is locally contained in an ideal of the form J/f , where J is a coherent ideal of \mathcal{O}_X , and we have $[I^{n+1} : I^n] = \frac{1}{f}[(fI)^{n+1} : (fI)^n]$.

Theorem 3.2.2. *Let X/S be an integral Noetherian scheme with sheaf of rational functions R_X . Let I be a coherent \mathcal{O}_X -submodule of R_X and $\pi : Bl_I(X) \rightarrow X$ be the blow-up of X over I .*

(1) $T_{X/S}(\hat{I}) \subset T_{X/S}^\pi \subset T_{X/S}(\bar{I})$ (the right inclusion holds when the normalisation morphism p is tamely ramified).

(2) $T_{X/S}(I) \subset T_{X/S}^\pi \cap T_{X/S}(\bar{I}) \cap T_{X/S}(\hat{I})$.

Assume that p is tamely ramified. If $\hat{I} = \bar{I}$ (e.g. $I = \bar{I}$), then $T_{X/S}^\pi = T_{X/S}(\bar{I})$. If \bar{I} is radical, then π is differentially ramified.

Proof. We first prove $T_{X/S}(I) \subset T_{X/S}^\pi$. A derivation that preserves I extends to a derivation of the Rees algebra $\oplus_{n \geq 0} I^n$, and therefore induces a derivation of each affine chart $B_{(f)} = \cup_{j \geq 0} I^j / f^j$, $f \in I$, of $Bl_I(X) = \text{Proj } \oplus_{n \geq 0} I^n$. Since in such a chart $\tilde{I} = BI = (fB)$, clearly $\partial(\tilde{I}) \subset \tilde{I}$.

$T_{X/S}^\pi \subset T_{X/S}(\bar{I})$: We have $T_{X/S}^\pi \subset T_{X/S}^{\pi \circ p}$ since p is tamely ramified (Th. 2.3.1). As $\overline{Bl}_I(X)$ is Krull, by Theorem 2.2.1, (L_3) , a liftable derivation ∂ lifts to a derivation $\tilde{\partial} \in T_{\overline{Bl}_I(X)/S}$ that preserves the ideal of the critical divisor. Since $\tilde{I} = I \mathcal{O}_{\overline{Bl}_I(X)}$ is a locally principal ideal defining the critical locus of $\pi \circ p$ it follows that $\tilde{\partial}(\tilde{I}) \subset \tilde{I}$. Therefore ∂ preserves $(\pi \circ p)_*(\tilde{I}) = \bar{I}$.

$T_{X/S}(\hat{I}) \subset T_{X/S}^\pi$: Let $\hat{\pi} : Bl_{\hat{I}}(X) \rightarrow X$ be the blow-up of \hat{I} . We then have

$$T_{X/S}(\hat{I}) \subset T_{X/S}^{\hat{\pi}} = T_{X/S}^\pi$$

where the latter equality follows since $\hat{I}^n = I^n$ when $n \gg 1$, so $Bl_{\hat{I}}(X) = Bl_I(X)$. This completes the proof of the assertions (1) and (2).

Assume $\hat{I} = \bar{I}$, which holds in particular when I is radical, then (1) implies $T_{X/S}^\pi = T_{X/S}(\bar{I})$, which in particular implies that π is differentially ramified when I is radical. \square

Corollary 3.2.3. *Let $\pi : \tilde{X}/S \rightarrow X/S$ be a tamely ramified blow-up of a reduced subscheme V/S of an integral Noetherian scheme X/S , such that \tilde{X} is Krull. If X/S is smooth at generic points of V , then π is uniformly ramified.*

Suppose s is a point in S with perfect residue field and that the fibre V_s is a smooth subscheme of X_s , then \tilde{X}_s is regular and hence Krull (normal).

Proof. For each generic point of the critical locus its image x is a generic point of $V \subset X$. Let k be the residue field of the image s of x in S . The canonical exact sequence $0 \rightarrow \mathfrak{m}_{X,x}/\mathfrak{m}_{X,x}^2 \rightarrow k_{X,x} \otimes \Omega_{X/S,x} \rightarrow \Omega_{k_{X,x}/k} \rightarrow 0$ is (split) exact since $k_{X,x}/k$ is formally smooth. Assume that $\{x_1, \dots, x_r\} \subset \mathfrak{m}_{X,x}$ induces a basis $\{\bar{x}_1, \dots, \bar{x}_r\}$ of $\mathfrak{m}_{X,x}/\mathfrak{m}_{X,x}^2$, so $\mathfrak{m}_{X,x} = (x_1, \dots, x_r)$, and select elements $\omega_i \in \Omega_{X/S,x}$ such that $\{x_1, \dots, x_r, \omega_1, \dots, \omega_s\}$ induces a basis $\{1 \otimes d_{X/S}(x_1), \dots, 1 \otimes d_{X/S}(x_r), 1 \otimes \omega_1, \dots, 1 \otimes \omega_s\}$ of $k_{X,x} \otimes \Omega_{X/S,x}$. Since $\Omega_{X/S,x}$ is a free $\mathcal{O}_{X,x}$ -module of finite type it follows that $\{d_{X/S}(x_1), \dots, d_{X/S}(x_r), \omega_1, \dots, \omega_s\}$ are free generators of $\Omega_{X/S,x}$; hence there exist derivations $\partial_i \in T_{X/S,x}$ satisfying $\partial_i(x_j) = \delta_{ij}$. Since V is reduced and π is tamely ramified it follows that π is differentially ramified (Th. 3.2.2), so the assertion can be concluded from Theorem 3.1.3, (2). \square

For an ideal I of a polynomial ring R over a field it is in general a laborious task to compute its integral closure \bar{I} [33], although for monomial ideals it is simply the monomial ideal defined by the convex hull of the exponent vectors of I . The Ratliff-Rush ideal associated with I seems even more difficult to compute (see [7, 28]); for instance, I am unaware of any simple description of \hat{I} for monomial I . Notice that to compute the liftable derivations $T_{X/S}^\pi$ for a blow-up using Theorem 2.2.1 we need to know that $Bl_I(X)$ is normal. This however is difficult to read off from I (a sufficient condition is that I^n be integrally closed for high n).

Example 3.2.4. Let $R = \mathbf{Q}[t^4, t^5, t^6, t^7]$ (singular ring) and $I = (t^4, t^5)$. Then $I^2 = (t^8, t^9, t^{10})$, $[I^2 : I] = I$, $[I^3 : I^2] = (t^4, t^5, t^6, t^7) = \mathfrak{m} = \hat{I}$. We have $T_{R/\mathbf{Q}}(I) = Rt\partial_t$ and $T_{R/\mathbf{Q}}(\hat{I}) = Rt\partial_t + Rt^2\partial_t$. Hence $T_{R/\mathbf{Q}}(\hat{I}) \neq T_{R/\mathbf{Q}}(I)$.

Example 3.2.5. Let $R = \mathbf{Q}[x, y]_{(x,y)}$ (regular ring), $I = (x^{10}, x^8y, xy^4, y^5)$. Then $\hat{I} = (x^{10}, y^5, xy^4, x^7y^2, x^6y^3, x^8y)$ ([7, 28]) and $\bar{I} = (x^{10}, x^8y, x^6y^2, x^4y^3, xy^4, y^5)$. Generators of the module of derivations of a monomial \mathfrak{m} -primary ideal $J = (x^{a_i}y^{b_i})_{i=0}^r$, $a_i > a_{i+1}$ are easily computed. Let $h = \max(a_i - a_{i+1})$ and $w = \max(b_{i+1} - b_i)$. Then $T_R(J) = Rx\partial_x + Ry\partial_y + Ry^h\partial_x + Rx^w\partial_y$. In particular, $T_{R/\mathbf{Q}}(I) = (x, y^3)\partial_x + (x^7, y)\partial_y$, $T_{R/\mathbf{Q}}(\hat{I}) = (x, y)\partial_x + (x^5, y)\partial_y$, and $T_{R/\mathbf{Q}}(\bar{I}) = (x, y)\partial_x + (x^3, y)\partial_y$, so $T_{R/\mathbf{Q}}(I) \neq T_{R/\mathbf{Q}}(\hat{I}) \neq T_{R/\mathbf{Q}}(\bar{I})$. Consider the morphism

$$\pi : Bl_I(\text{Spec } R) \rightarrow \text{Spec } R.$$

Since $\hat{I} \neq \bar{I}$ it follows that $Bl_I(\text{Spec } R)$ cannot be normal, so we cannot use Theorem 2.2.1 to compute $T_{R/\mathbf{Q}}^\pi$; we know only that $(x, y)\partial_x + (x^5, y)\partial_y \subset T_{R/\mathbf{Q}}^\pi \subset (x, y)\partial_x + (x^3, y)\partial_y$. It would require some effort to determine whether the derivations $x^3\partial_y, x^4\partial_y$ belong to $T_{R/\mathbf{Q}}^\pi$.

Remark 3.2.6. We have $T_{X/S}(I) \subset T_{X/S}(I^n) \subset T_{X/S}([I^n : I^{n-1}])$, but in general $T_{X/S}(I^n) \not\subset T_{X/S}(I)$, when $I \neq \hat{I}$, as seen from *Example 3.2.4*, for $n = 3$.

We have $T_{X/S}(I) \subset T_{X/S}(\hat{I}) \subset T_{X/S}(\bar{I})$ where the first inclusion follows from *Remark 3.2.6*, while the latter, valid when p is tamely ramified, follows from *Theorem 3.2.2*; that these inclusions in general are strict is shown in *Example 3.2.5*. One can refine these inclusions. In [31] Shah defines for \mathfrak{m} -primary ideals in a local ring the notion of coefficient ideals, using coefficients of the Hilbert polynomial of I ; this notion was extended in [16, Def 3.21] to general ideals I in a Noetherian domain, defining coefficient ideals as contractions of pull-backs of I by certain birational morphisms. Let I be a coherent ideal of \mathcal{O}_X , where X is an integral and Noetherian scheme of dimension d , and let $I_{\{k\}}$, $k = 0, \dots, d$ be the coefficient ideals of I as defined in [loc. cit.]; put also $I_{\{d+1\}} = I$. One has $I \subset I_{\{d\}} = \hat{I} \subset I_{\{d-1\}} \subset \dots \subset I_{\{1\}} \subset I_{\{0\}} = \bar{I}$ [loc. cit] and one can prove, as in the beginning of the proof of *Theorem 3.2.2*, noting that the lifted derivations evidently preserve the intersections of local rings described in [loc. cit, 3.14], that $T_{X/S}(I) \subset T_{X/S}(I_{\{d\}}) = T_{X/S}(\hat{I}) \subset T_{X/S}(I_{\{d-1\}}) \subset \dots \subset T_{X/S}(I_{\{k\}}) \subset \dots \subset T_{X/S}(I_{\{1\}}) \subset T_{X/S}(\bar{I})$, at least if the birational map that is used to define $I_{\{k\}}$ is tamely ramified. Let $J_{\{k\}}$ be the smallest coherent ideal that contains $I_{\{k\}}$ such that $T_{X/S}(I_{\{k-1\}}) \subset T_{X/S}(J_{\{k\}})$. We then have

$$I_{\{k\}} \subset J_{\{k\}} \subset I_{\{k-1\}},$$

and one may wonder when these inclusions are strict? In *Examples 3.2.6* and *3.2.5* we have $J_{\{d+1\}} = I_{\{d\}} (= \hat{I})$. If one knows a priori that $J_{\{k\}} = I_{\{k-1\}}$, one can compute $I_{\{k-1\}}$ from a knowledge of $I_{\{k\}}$ and $T_{X/S}(I_{\{k-1\}})$, since $J_{\{k\}} = \mathcal{D}_{X/S}(T_{X/S}(I_{\{k-1\}}))I_{\{k\}}$ and the ring $\mathcal{D}_{X/S}(T_{X/S}(I_{\{k-1\}}))$ of differential operators generated by $T_{X/S}(I_{\{k-1\}})$. However, one cannot use *Theorem 2.2.1* to describe $T_{X/S}(I_{\{k-1\}})$, since the source of the birational morphisms used to define $I_{\{k-1\}}$ is not Krull.

We end this paper with two concrete illustrations of the use of *Theorems 2.1.1* and *3.2.2*.

Example 3.2.7. Let A be the polynomial ring $k[y_1, \dots, y_r]$ localised at some prime ideal, and assume that $A \subset R$ is tamely ramified and $K(R)/K(A)$ is finite, where R is a discrete valuation ring with valuation ν . *Theorem 2.1.1* implies $\partial = a_1\partial_{y_1} + \dots + a_r\partial_{y_r} \in T_{A/k}^\pi$, $a_i \in A$, if and only if $\nu(a_i) \geq \nu(y_i)$, $i = 1, \dots, r$. To be very explicit, let $A = k[y_1, y_2]_{(y_1, y_2)}$, $R = k[x_1, x_2]_{(x_1)}$, so $\mathfrak{m}_A = (y_1, y_2)$ and $\mathfrak{m}_R = (x_1)$, and define a birational map $\pi : A \rightarrow R$ by $y_1 = x_1$, $y_2 = x_1^n x_2$ where n is an integer ≥ 1 . Then $\partial_{y_1} = \partial_{x_1} - nx_2/x_1\partial_{x_2}$ and $\partial_{y_2} = x_1^{-n}\partial_{x_2}$, acting on $K(R) = k(x_1, x_2)$. The condition $\partial \in T_{A/k}(\log^\pi \mathfrak{m}_A)$ is that $\partial \in T_{A/k}$ and $\nu(\partial(y_1)) \geq \nu(y_1) = 1$, $\nu(\partial(y_2)) \geq \nu(y_2)$, i.e. $a_1 \in (y_1, y_2)$ and $a_2 \in (y_1^n, y_2)$; therefore $T_{A/k}(\log^\pi \mathfrak{m}_A) = \mathfrak{m}_A\partial_{y_1} + (y_1^n, y_2)\partial_{y_2}$. If $\text{Char } k = 0$ or n is coprime to $\text{Char } k$, by *Theorem 2.1.1*

$$T_{A/k}^\pi = T_{A/k}(\log^\pi \mathfrak{m}_A) = \mathfrak{m}_A\partial_{y_1} + (y_1^n, y_2)\partial_{y_2}.$$

We have then

$$RT_{A/k}^\pi = R\nabla_1 + R\partial_{x_2} = T_{R/k}(I_{C_\pi}),$$

where the critical ideal I_{C_π} is (x_1) . If $n = 1$, so π comes by a chart of the blow-up of \mathfrak{m}_A , then $T_{A/k}^\pi = T_{A/k}(\log^\pi \mathfrak{m}_A) = \mathfrak{m}_A\partial_{y_1} + \mathfrak{m}_A\partial_{y_2} = T_{A/k}(\mathfrak{m}_A)$, so π is differentially ramified, in agreement with *Theorem 3.2.2*, noting that \mathfrak{m}_A is radical. If $n \geq 2$, corresponding to a successive blowing up of points, then the derivations $y_1^k\partial_{y_2} \in T_{A/k}(\mathfrak{m}_A)$, $k = 1, \dots, n-1$ are not liftable. In this case $A \rightarrow R$ comes by a chart of the blow-up of the ideal $I = (y_1^n, y_2)$; this ideal is Ratliff-Rush and integrally closed $I = \hat{I} = \bar{I}$. It is clear that $T_{A/k}^\pi = \mathfrak{m}_A\partial_{y_1} + (y_1^n, y_2)\partial_{y_2} = T_{A/k}(I)$,

as predicted by Theorem 3.2.2. If $\text{Char } k = p > 0$ and p divides n , so π has wild ramification, then $T_{A/k}^\pi = A\partial_{y_1} + (y_1^n, y_2)\partial_{y_2}$.

Example 3.2.8. Put $B = \mathbf{Q}[x, s, t]/(s^4 - xt)$ and $A = \mathbf{Q}[x, y]$ and define the homomorphism $\pi : A \rightarrow B$, $\pi(x) = x$, $\pi(y) = x^2s$. The critical divisor is given by the principal ideal $(s) \subset B$ and $x = s^4/t$ in $B_{(s)}$, so the corresponding valuation $\nu : A \rightarrow \mathbf{Z}$ is determined by $\nu(x) = 4$, $\nu(y) = 9$. A derivation $a\partial_x + b\partial_y \in T_{A/\mathbf{Q}}$ lifts to a derivation of B if $\nu(a) \geq \nu(x) = 4$ and $\nu(b) \geq \nu(y) = 9$ (Th. 2.1.1), noting that B is a normal ring, implying $T_{A/\mathbf{Q}}^\pi = (x, y)\partial_x + (x^2, y)\partial_y$.

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